Phys 487 Discussion 3 – Angular Momentum Addition : Clebsch-Gordan Coefficients

We are in the process of learning how to add two angular momenta, j_1 and j_2. We can add them in two ways:

- we can list all the combinations $|m_1 m_2\rangle$ that are allowed by $|m_i| \le j_i$ ….. or ….
- we can list all the eigenstates $|JM\rangle$ of the TOTAL ang. mom. operators $J^2 \equiv \left|\vec{j}_1 + \vec{j}_2\right|^2$ & $J_z \equiv \left(\vec{j}_1 + \vec{j}_2\right)\cdot\hat{z}$.

The eigenstates $|m_1 m_2\rangle$ and $|JM\rangle$ provide different bases with which to describe the sum of one angular momentum j_1 with another one j_2. In class, we learned how to read tables of Clebsch-Gordan Coefficients (back page) to express a $|JM\rangle$ as a linear combination of $|m_1 m_2\rangle$'s. We also wrote down the procedure for calculating the coefficients but didn't try it out yet.

Problem 1 : Deuterium Atom

We'll continue to explore our example from class: $j_1 = 1$ is the spin s_d of a deuterium nucleus and $j_2 = ½$ is the spin s_e of an electron. If we bring the nucleus and electron together[1] we get a deuterium atom with total spin $j_1 \oplus j_2$, where the $\oplus$ symbol means "this addition is a more involved procedure than 2+3 = 5!".

(a) Make a plot with the m_1 axis pointing upward and the m_2 axis pointing sideways, and mark with solid circles all the points (m_1,m_2) where $|m_1 m_2\rangle$ is a physically-possible state. How many states do you have?

(b) Make a plot with the J axis pointing upward and the M axis pointing sideways, and mark with solid circles all the points (J, M) where $|JM\rangle$ is a physically possible state. **Leave lots of space between your tick marks on the horizontal (*M*) axis** … like an inch of space … we're going to write things under these tick marks. To make your plot, remember the angular momentum addition rule:

- the total J quantum number runs from $|j_1 - j_2|$ to $(j_1 + j_2)$ in steps of 1,

and the general rule for any angular momentum:

- the m (or M, or m_2, …) quantum number runs from $-j$ to $+j$ (or $-J$ to $+J$, or $-j_2$ to $+j_2$,…)

How many states do you have? It should be the same as in part (a)!

(c) **Underneath** each of the M tick marks that has at least one solid circle above it — i.e. under each M value that has at least one valid $|JM\rangle$ eigenstate — write a list of all the $|m_1 m_2\rangle$ states that might contribute to said M value. To do this, you must realize that $m_1 + m_2 = M$.

(d) As we discussed, $m_1 + m_2 = M$ … but $j_1 + j_2 \neq J$. The m quantum number is additive, but the j quantum number is not. Do you understand why this is? If not, check the hint[2] ... then ask ask ask!!!

(e) Consult the attached Clebsch-Gordan page and write the eigenstate $|J,M\rangle = \left|\tfrac{1}{2},-\tfrac{1}{2}\right\rangle_{J,M}$ as a linear combination of $|m_1,m_2\rangle$ eigenstates.

(f) Consult nothing and write the $\left|\tfrac{3}{2},-\tfrac{3}{2}\right\rangle_{J,M}$ eigenstate as a linear combination of $|m_1,m_2\rangle$ eigenstates.

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[1] To be exact, we are making an atom with zero *orbital* angular momentum, i.e. with $l = 0$! We are adding two spins here: j_1 = spin of nucleus and j_2 = spin of electron; we are not ready yet to add in a *third* source of angular momentum $j_3 = l$ of electron around nucleus!.

[2] j and m are **quantum numbers** that **label** the **eigenvalues** of certain operators … but they **are not** eigenvalues. A similar case from our old 1D problems is the **energy quantum number** n : it is an integer that is used to **label energy eigenvalues** but it is not an energy itself! We derived formulae like $E_n = -13.6\text{ eV} / n^2$ to translate the quantum number n into the corresponding eigenvalue E_n.

(g) OK, you just realized that the $|J,M\rangle$ states $\left|\frac{3}{2},\pm\frac{3}{2}\right\rangle_{J,M}$ each match a single $|m_1,m_2\rangle$ state: $\left|\pm1,\pm\frac{1}{2}\right\rangle_{m_1,m_2}$.

These are called "stretched" states because all *j* vectors are <u>maximally aligned</u> along or against the *z* axis.

The stretched states should be found at the top-right and top-left corners of the (*M*, *J*) plot you have been making. Circle the one with positive *M*. This will be the **starting point** of our Clebsch-Gordan calculations as it is always <u>uniquely determined</u> : $|J_{\max}, M_{\max}\rangle = |m_{1\max}, m_{2\max}\rangle = |+j_1, +j_2\rangle$. Simple starting point!

(h) THE WHOLE THING: As we saw in lecture 3A last blackboard, here is the **procedure** for building *all* the $|J, M\rangle$ states from $|m_1, m_2\rangle$ states, and thereby calculating all the Clebsch-Gordan (CG) coefficients:

A. Build the stretched state $|J_{\max}, M_{\max}\rangle$ = top-right point on your plot.

B. Build the states with the same *J* but lower *M* — step to the LEFT on your plot — by applying the **step-down** operator J_-. Recall :

$\hat{J}_\pm|j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\;|j,m\pm1\rangle$ → apply J_- to the $|J, M\rangle$ form

$\hat{Q}_{1+2} = \hat{Q}_{1\text{only}} + \hat{Q}_{2\text{only}}$ → apply $J_- = J_{1-} + J_{2-}$ to the $|m_1, m_2\rangle$

Step-down until you have filled in the whole line with the *J* you are currently working on. Check the CG tables at the back → are your answers correct?

C. Go back to the right-hand-side of the *J* line you've just completed, move one step to the left, to $|J, J-1\rangle_{J,M}$, then move DOWN one step and build $|J-1, J-1\rangle_{J,M}$ using **orthonormality**.

If you look at the $|m_1,m_2\rangle$ states you wrote below this *M* column, you'll see that you have only one state left to build, and it has to be orthonormal to all the ones above it, and you have those already! Get the idea? Great!

Finally repeat B and C until all the states $|J, M\rangle$ are written as linear combinations of the states $|m_1, m_2\rangle$.

Problem 2 → Homework 3 Problem 1 : Repeat for 3/2 x 1

Add a spin-3/2 particle and a spin-1 particle to build the state $|J, M\rangle = \left|\frac{3}{2},+\frac{1}{2}\right\rangle_{J,M}$ from $|m_1, m_2\rangle$ states.

Calculate the result using the A, B, C method above, doing only the steps you need (!) to reach the state you want. Check your answer against the CG table at the end, of course.