Discussion 1

The collection of *all* functions of x constitutes a vector space, but for our purposes it is much too large. To represent a possible physical state, the wave function Ψ must be *normalized*:

$$\int |\Psi|^2 \, dx = 1.$$

The set of all **square-integrable functions**, on a specified interval,[2]

$$f(x) \quad \text{such that} \quad \int_a^b |f(x)|^2 \, dx < \infty. \qquad [3.4]$$

constitutes a (much smaller) vector space (see Problem 3.1(a)). Mathematicians call it $L_2(a, b)$; physicists call it **Hilbert space.**[3] In quantum mechanics, then,

$$\boxed{\textbf{Wave functions live in Hilbert space.}} \qquad [3.5]$$

We define the **inner product of two functions**, $f(x)$ and $g(x)$, as follows:

$$\langle f|g\rangle \equiv \int_a^b f(x)^* g(x) \, dx. \qquad [3.6]$$

If f and g are both square-integrable (that is, if they are both in Hilbert space), their inner product is guaranteed to exist (the integral in Equation 3.6 converges to a finite number).[4] This follows from the integral **Schwarz inequality**:[5]

$$\left| \int_a^b f(x)^* g(x) \, dx \right| \le \sqrt{\int_a^b |f(x)|^2 \, dx \int_a^b |g(x)|^2 \, dx}. \qquad [3.7]$$

Problem 3.1

(a) Show that the set of all square-integrable functions is a vector space (refer to Section A.1 for the definition). *Hint:* The main problem is to show that the sum of two square-integrable functions is itself square-integrable. Use Equation 3.7. Is the set of all *normalized* functions a vector space?

(b) Show that the integral in Equation 3.6 satisfies the conditions for an inner product (Section A.2).

$$\langle\beta|\alpha\rangle = \langle\alpha|\beta\rangle^*, \tag{A.19}$$

$$\langle\alpha|\alpha\rangle \geq 0, \quad \text{and} \quad \langle\alpha|\alpha\rangle = 0 \Leftrightarrow |\alpha\rangle = |0\rangle, \tag{A.20}$$

$$\langle\alpha|(b|\beta\rangle + c|\gamma\rangle) = b\langle\alpha|\beta\rangle + c\langle\alpha|\gamma\rangle. \tag{A.21}$$

Now, the outcome of a measurement has got to be *real*, and so, *a fortiori*, is the *average* of many measurements:

$$\langle Q \rangle = \langle Q \rangle^*. \qquad [3.14]$$

But the complex conjugate of an inner product reverses the order (Equation 3.8), so

$$\langle \Psi | \hat{Q}\Psi \rangle = \langle \hat{Q}\Psi | \Psi \rangle. \qquad [3.15]$$

and this must hold true for any wave function Ψ. Thus operators representing *observables* have the very special property that

$$\langle f | \hat{Q} f \rangle = \langle \hat{Q} f | f \rangle \quad \text{for all } f(x). \qquad [3.16]$$

We call such operators **hermitian**.

Actually, most books require an ostensibly stronger condition:

$$\langle f | \hat{Q} g \rangle = \langle \hat{Q} f | g \rangle \quad \text{for all } f(x) \text{ and all } g(x). \qquad [3.17]$$

But it turns out, in spite of appearances, that this is perfectly equivalent to my definition (Equation 3.16), as you will prove in Problem 3.3. So use whichever you like. The essential point is that a hermitian operator can be applied either to the first member of an inner product or to the second, with the same result, and hermitian operators naturally arise in quantum mechanics because their expectation values are real:

Observables are represented by hermitian operators. [3.18]

Problem 3.3 Show that if $\langle h|\hat{Q}h\rangle = \langle \hat{Q}h|h\rangle$ for all functions h (in Hilbert space), then $\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle$ for all f and g (i.e., the two definitions of "hermitian"—Equations 3.16 and 3.17—are equivalent). *Hint:* First let $h = f + g$, and then let $h = f + ig$.

Problem 3.4

(a) Show that the *sum* of two hermitian operators is hermitian.

(b) Suppose $\hat{Q}$ is hermitian, and α is a complex number. Under what condition (on α) is $\alpha\hat{Q}$ hermitian?

(c) When is the *product* of two hermitian operators hermitian?

(d) Show that the position operator ($\hat{x} = x$) and the hamiltonian operator ($\hat{H} = -(\hbar^2/2m)d^2/dx^2 + V(x)$) are hermitian.

Problem 3.5 The **hermitian conjugate** (or **adjoint**) of an operator $\hat{Q}$ is the operator $\hat{Q}^\dagger$ such that

$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}^\dagger f|g\rangle \quad \text{(for all } f \text{ and } g\text{).} \tag{3.20}$$

(A hermitian operator, then, is equal to its hermitian conjugate: $\hat{Q} = \hat{Q}^\dagger$.)

(a) Find the hermitian conjugates of x, i, and d/dx.

(c) Show that $(\hat{Q}\hat{R})^\dagger = \hat{R}^\dagger\hat{Q}^\dagger$.

Problem 3.2

(a) For what range of ν is the function $f(x) = x^\nu$ in Hilbert space, on the interval (0, 1)? Assume ν is real, but not necessarily positive.

(b) For the specific case $\nu = 1/2$, is $f(x)$ in this Hilbert space? What about $xf(x)$? How about $(d/dx)f(x)$?

5.23 Algebra of Angular Momentum (Stony Brook)

Given the commutator algebra

$$[J_1, J_2] = iJ_3$$

$$[J_2, J_3] = iJ_1$$

$$[J_3, J_1] = iJ_2$$

a) Show that $J^2 = J_1^2 + J_2^2 + J_3^2$ commutes with J_3.

b) Derive the spectrum of $\{J^2, J_3\}$ from the commutation relations.