The Meaning of Grad, Div, Curl, and the GGS Theorem

Here is a recap of the physical meaning of the differential operations **gradient**, **divergence**, and **curl** and of the **Gauss-Green-Stokes** theorem.

- **GRAD**: $\vec{\nabla} f(\vec{r})$ tells you the **uphill slope** of *f*: it points in the *direction of maximum increase* of *f* at each point $\vec{r}$ ("uphill") , and its magnitude gives you the slope in that direction.
  Why? Grad is built from slopes, i.e. derivatives: $\vec{\nabla} f = \hat{x}\,\partial_x f + \hat{y}\,\partial_y f + \hat{z}\,\partial_z f$
  The biggest slopes get the biggest weight, and that "turns" the gradient vector in the direction of greatest net slope.
  Key physics example: $\vec{F} = -\vec{\nabla} U$

- **DIV**: $\vec{\nabla}\cdot\vec{E}(\vec{r})$ tells you the **outward flux** of the field $\vec{E}$ emanating from each point $\vec{r}$. Put differently, a positive [negative] divergence at a given location tells you the **density of point sources** [sinks] from which the field emanates [into which it sinks] at that location.
  Why? Div is built from the "diagonal" partial derivatives $\partial_x E_x$, $\partial_y E_y$, $\partial_z E_z$. When these are positive, the field *grows in the direction it points* → outward flux!
  Key physics example: $\vec{\nabla}\cdot\vec{E} = \rho / \varepsilon_0$

- **CURL**: $\vec{\nabla}\times\vec{E}(\vec{r})$ tells you the *axis* around which the field *twists maximally*; if $\vec{E}$ is a force field and you place an object at $\vec{r}$, the curl is proportional to the **torque** on that object.
  Why? Curl is built from the "off-diagonal" partial derivatives $\partial_i E_j$ where $i \neq j$.
  Imagine a little sphere around a point $\vec{r}$; when these derivatives are non-zero at $\vec{r}$, we have an *imbalance in tangential field components* at the sphere's surface *on either side of* $\vec{r}$ … which is precisely what creates torque.
  Key physics example: $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$

The **Gauss-Green-Stokes** theorem ("**GGS**" for short) is a collection of three integral relations that involve grad, div, and curl and serve to *quantify* the physical meaning of those operations: they incorporate the qualitative interpretations we made above above into rigorous formulae. In mathematics, GGS is a subset of a whole class of theorems that all have the form shown at right. As usual, $\mathbb{R}$ denotes a **region** of space; the symbol $\partial\mathbb{R}$ denotes the **boundary** of that region. The symbols *f* and *df* denote respectively "some field *f* " and "some derivative of that field *f* ". Grad, div, and curl each appear in one of the many incarnations of this class, giving us the three theorems we call GGS. They are the three instances that appear most commonly in physics:

$$\int_{\mathbb{R}} df = \int_{\partial\mathbb{R}} f$$

$\mathbb{R}^1$: Gradient Theorem	$\mathbb{R}^2$: Stokes' Theorem	$\mathbb{R}^3$: Gauss' Theorem
$\int_{\vec{a}}^{\vec{b}} \vec{\nabla} f \cdot d\vec{l} = f(\vec{b}) - f(\vec{a})$	$\int_{\text{Surf}} (\vec{\nabla}\times\vec{E})\cdot d\vec{A} = \oint_{\partial\text{Surf}} \vec{E}\cdot d\vec{l}$	$\int_{\text{Vol}} (\vec{\nabla}\cdot\vec{E})\, dV = \oint_{\partial\text{Vol}} \vec{E}\cdot d\vec{A}$

Grad, div, and curl appear on the left-hand sides of these relations. Look closely at these expressions; you will see that the integrals on the right-hand sides *exactly* encode the concepts of "uphill slope" for $\vec{\nabla} f$, "torque" – via work around a closed loop – for $\vec{\nabla}\times\vec{E}$, and "outward flux" for $\vec{\nabla}\cdot\vec{E}$. ☺☺☺