1D SHO $\hat{H}(x)=\dfrac{1}{2m}\left(\hat{p}^2+m^2\omega^2x^2\right)$ Define $x_0\equiv\sqrt{\dfrac{\hbar}{m\omega}}$, $\xi\equiv\dfrac{x}{x_0}$ $\rightarrow$ $\hat{H}(\xi)=\dfrac{\hbar\omega}{2}\left(\xi^2-\dfrac{d^2}{d\xi^2}\right)$

$$E_n=\left(n+\tfrac{1}{2}\right)\hbar\omega\ ,\quad \psi_n(x)=\left(\frac{1}{\pi x_0^2}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n\left(\frac{x}{x_0}\right)e^{-\frac{x^2}{2x_0^2}}\quad\text{where}$$

$$H_0(\xi)=1,\quad H_2(\xi)=4\xi^2-2,$$
$$H_1(\xi)=2\xi,\ H_3(\xi)=8\xi^3-12\xi,$$
$$H_n(\xi)=(-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}e^{-\xi^2}$$

$$\hat{a}_\pm=\frac{1}{\sqrt{2}}\left(\xi\mp\frac{d}{d\xi}\right)\rightarrow\begin{matrix}\hat{a}_+\psi_n=\sqrt{n+1}\,\psi_{n+1}\\ \hat{a}_-\psi_n=\sqrt{n}\,\psi_{n-1}\end{matrix},\quad \begin{matrix}\hat{H}=\hbar\omega\left(\hat{a}_+\hat{a}_-+\tfrac{1}{2}\right)\\ \left[\hat{a}_-,\hat{a}_+\right]=1\end{matrix}$$

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

	J	J	...
	M	M	...
m_1 m_2			
m_1 m_2	Coefficients		
. .			

$$Y_1^0=\sqrt{\frac{3}{4\pi}}\cos\theta$$
$$Y_1^1=-\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{i\phi}$$
$$Y_2^0=\sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)$$
$$Y_2^1=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\,e^{i\phi}$$
$$Y_2^2=\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\,e^{2i\phi}$$
$$Y_\ell^{-m}=(-1)^m Y_\ell^{m*}$$

$$\langle j_1j_2m_1m_2|j_1j_2JM\rangle=(-1)^{J-j_1-j_2}\langle j_2j_1m_2m_1|j_2j_1JM\rangle$$

1/2×1/2

m_1 m_2	J=1, M=+1
+1/2 +1/2	1

m_1 m_2	J=1, M=0	J=0, M=0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	J=1, M=−1
−1/2 −1/2	1

1×1/2

m_1 m_2	J=3/2, M=+3/2
+1 +1/2	1

m_1 m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	J=3/2, M=−3/2
−1 −1/2	1

2×1

m_1 m_2	J=3, M=+3
+2 +1	1

m_1 m_2	J=3, M=+2	J=2, M=+2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	2/5	−1/2	1/10

m_1 m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0 −1	2/5	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	J=3, M=−2	J=2, M=−2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	J=3, M=−3
−2 −1	1

1×1

m_1 m_2	J=2, M=+2
+1 +1	1

m_1 m_2	J=2, M=+1	J=1, M=+1
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	J=2, M=−1	J=1, M=−1
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	J=2, M=−2
−1 −1	1

2×1/2

m_1 m_2	J=5/2, M=+5/2
+2 +1/2	1

m_1 m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	J=5/2, M=−5/2
−2 −1/2	1

3/2×1/2

m_1 m_2	J=2, M=+2
+3/2 +1/2	1

m_1 m_2	J=2, M=+1	J=1, M=+1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	J=2, M=0	J=1, M=0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	J=2, M=−1	J=1, M=−1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	J=2, M=−2
−3/2 −1/2	1

3/2×1

m_1 m_2	J=5/2, M=+5/2
+3/2 +1	1

m_1 m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	J=5/2, M=−5/2
−3/2 −1	1

3/2×3/2

m_1 m_2	J=3, M=+3
+3/2 +3/2	1

m_1 m_2	J=3, M=+2	J=2, M=+2
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m_1 m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m_1 m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m_1 m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m_1 m_2	J=3, M=−2	J=2, M=−2
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m_1 m_2	J=3, M=−3
−3/2 −3/2	1

2×3/2

m_1 m_2	J=7/2, M=+7/2
+2 +3/2	1

m_1 m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m_1 m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

m_1 m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

m_1 m_2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

m_1 m_2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

m_1 m_2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m_1 m_2	J=7/2, M=−7/2
−2 −3/2	1

2×2

m_1 m_2	J=4, M=+4
+2 +2	1

m_1 m_2	J=4, M=+3	J=3, M=+3
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m_1 m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

m_1 m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

m_1 m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

m_1 m_2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

m_1 m_2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m_1 m_2	J=4, M=−3	J=3, M=−3
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m_1 m_2	J=4, M=−4
−2 −2	1

Generators of Transformations (bolded)

spatial translation $U_{\hat{a}}(\Delta)=\exp\left(\Delta\hat{a}\cdot\vec{\nabla}\right)=\exp\left(-\dfrac{\Delta\hat{a}}{i\hbar}\cdot\mathbf{p}\right)$

rotation around $\hat{z}$ $U_{\hat{z}}(\Delta)=\exp\left(\Delta\hat{z}\cdot\dfrac{\partial}{\partial\phi}\right)=\exp\left(-\dfrac{\Delta\hat{z}}{i\hbar}\cdot\mathbf{L_z}\right)$

time translation $U(\Delta)=\exp\left(\Delta\dfrac{\partial}{\partial t}\right)=\exp\left(-\dfrac{\Delta}{i\hbar}\cdot(\mathbf{-H})\right)$

Variational Principle $E_{gs} \le \langle\psi|H|\psi\rangle$ for any ψ

WKB Approximation In **A**llowed ($V<E$) & **B**locked ($V>E$) regions, with $p(x) \equiv \sqrt{2m(E-V(x))}$,

Solution forms : $\psi_{\mathbf{A}}(x) = \frac{A}{\sqrt{p(x)}}\exp\left[\pm i\int^{x}\frac{p(x')}{\hbar}dx'\right]$, $\psi_{\mathbf{B}}(x) = \frac{B}{\sqrt{|p(x)|}}\exp\left[\pm\int^{x}\frac{|p(x')|}{\hbar}dx'\right]$

Connection formulae at turning points $x=a$: with $\int k \equiv \int p(x')/\hbar\ dx'$ & "barrier" ≡ $V>E$ region

barrier on LEFT ($x<a$) :

$$\psi_{\mathbf{B}}(x) = \frac{1}{2}\frac{C}{\sqrt{|p|}}\exp\left[-\int_x^a |k|\right] \quad \text{matches to} \quad \psi_{\mathbf{A}}(x) = \frac{C}{\sqrt{p}}\cos\left[\int_a^x k - \frac{\pi}{4}\right]$$

$$\psi_{\mathbf{B}}(x) = \frac{1}{2}\frac{D}{\sqrt{|p|}}\exp\left[+\int_x^a |k|\right] \quad \text{matches to} \quad \psi_{\mathbf{A}}(x) = \frac{D}{\sqrt{p}}\cos\left[\int_a^x k + \frac{\pi}{4}\right]$$

barrier on RIGHT ($x>a$) :

$$\psi_{\mathbf{B}}(x) = \frac{1}{2}\frac{C}{\sqrt{|p|}}\exp\left[-\int_a^x |k|\right] \quad \text{matches to} \quad \psi_{\mathbf{A}}(x) = \frac{C}{\sqrt{p}}\cos\left[\int_x^a k - \frac{\pi}{4}\right]$$

$$\psi_{\mathbf{B}}(x) = \frac{1}{2}\frac{D}{\sqrt{|p|}}\exp\left[+\int_a^x |k|\right] \quad \text{matches to} \quad \psi_{\mathbf{A}}(x) = \frac{D}{\sqrt{p}}\cos\left[\int_x^a k + \frac{\pi}{4}\right]$$

"Barrier on **right**" formulae are **IDENTICAL** to "barrier on **left**" ones except that the **order of the integral bounds is reversed**. In ALL cases, the lower bound is at smaller x than the upper bound.

Perturbation Theory – Time-Independent $H = H_0 + H'$

- H_0 solvable w eigen-* $\{E_n^{(0)}\},\{|n^{(0)}\rangle\}$
- H' is perturbation $\ll H_0$

Expansions for eigen-* of H : $E_n = E_n^{(0)} + E_n^{(1)} + \cdots$ & $|n\rangle = |n^{(0)}\rangle + |n^{(1)}\rangle + \cdots$

For a **non-degenerate** eigenvalue $E_n^{(0)}$ of H_0 : $|n^{(1)}\rangle = \sum_{m\neq n}\frac{\langle m^{(0)}|H'|n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}}|m^{(0)}\rangle$,

$$E_n^{(1)} = \langle n^{(0)}|H'|n^{(0)}\rangle, \quad E_n^{(2)} = \langle n^{(0)}|H'|n^{(1)}\rangle = \sum_{m\neq n}\frac{|\langle m^{(0)}|H'|n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}, \quad \ldots \quad E_n^{(j)} = \langle n^{(0)}|H'|n^{(j-1)}\rangle$$

For a **degenerate** eigenvalue $E_D^{(0)}$ of H_0 :

- Let $\{|\alpha_1^{(0)}\rangle, \ldots, |\alpha_n^{(0)}\rangle\}$ = degenerate subspace D sharing eigenvalue $E_D^{(0)}$
- Find $\{|\beta_1^{(0)}\rangle, \ldots, |\beta_n^{(0)}\rangle\}$ = eigenvectors of H' within subspace D

  = linear combinations of $|\alpha_i^{(0)}\rangle$ states that diagonalize $\mathbf{H}'$

→ 1st order energy correction is $E_{\beta_i}^{(1)} = \langle\beta_i^{(0)}|H'|\beta_i^{(0)}\rangle$