Physics 486 – Homework #10

In the next lecture, we will discuss the **probability current density**

\[
\mathbf{j}(\mathbf{r},t) = \text{Re} \left[ \Psi^* \frac{\hat{p}}{m} \Psi \right] = \text{Re} \left[ \Psi^* \frac{\hbar \hat{\nabla}}{im} \Psi \right].
\]

It has units of probability / (time · area) = probability-current / area, and is related to the **probability density** \(\rho(\mathbf{r},t) = \Psi^* \Psi\) by the **continuity equation**:

\[
-\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{j}.
\]

This celebrated equation expresses conservation of probability by saying “if the probability density \(\rho\) drops at a certain point, it must have flowed away from that point in the form of a current \(\mathbf{j}\)”. The probability current density takes on a very simple form for a momentum eigenstate \(\Psi_{p_0}\) of definite momentum \(p_0\) and speed \(\vec{v}_0 = \frac{p_0}{m}\):

\[
\mathbf{j}(\mathbf{r},t) = \rho(\mathbf{r},t) \vec{v}_0
\]

for a momentum eigenstate.

This simple and familiar relation helps us to visualize what \(\mathbf{j}\) is, but is only true for a momentum eigenstate.

**Problem 1: Transmission Coefficient & Probability Current** adapted from Gr 2.34

Consider the step potential

\[
V(x,y,z) = \begin{cases} 
0 & \text{if } z \leq 0 \\
V_0 & \text{if } z > 0 
\end{cases}
\]

(a) The true definitions of the **reflection and transmission coefficients** at a potential boundary are ratios of the probability current densities associated with the incident, reflected, and transmitted waves:

\[
R \equiv \frac{\mathbf{j}_{\text{reflected}} \cdot \vec{A}}{\mathbf{j}_{\text{incident}} \cdot \vec{A}} \quad \text{and} \quad T \equiv \frac{\mathbf{j}_{\text{transmitted}} \cdot \vec{A}}{\mathbf{j}_{\text{incident}} \cdot \vec{A}} \quad \text{where } \vec{A} \text{ is the area vector perpendicular to the boundary.}
\]

Send in an incident plane wave of energy \(E \geq V_0\) from \(z = -\infty\) : \(\psi_{\text{in}}(z) = Ae^{ikz}\) where \(k = \sqrt{2mE/\hbar}\).

As you know, this will produce a reflected wave \(\psi_{\text{re}}(z) = Be^{-ikz}\) and transmitted wave \(\psi_{\text{tr}}(z) = Ce^{ikz}\).

Calculate \(K\) of the transmitted wave in terms of \(E, m,\) and \(V_0,\) and the amplitudes \(B\) & \(C\) in terms of \(A, k,\) and \(K\).

(b) Calculate the probability currents \(\mathbf{j} = \text{Re} [\Psi^* \frac{\hbar \hat{\nabla}}{(im)} \Psi]\) for the incident, reflected, and transmitted waves in terms of the amplitudes \(A, B, C,\) the wavenumbers \(k, K,\) and physical constants. Remember that your results should be vectors.

(c) Calculate the reflection and transmission coefficients \(R\) and \(T\) in terms of \(k\) and \(K.\)