PHY460: Problem Set #1 Due: Wednesday 9/20 by 5pm in Yellow PHY460 homework box between Loomis and MRL on 2nd Floor

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Please look over the homework and grading policies in the syllabus before you begin.

1. Let us define an orthonormal basis of quantum states $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle\}$

   (a) Write down the value of the inner product $\langle\psi_i|\psi_j\rangle$ for $i, j = 1, 2, 3, 4$.

   (b) Suppose that we have an operator $A = i|\psi_2\rangle\langle\psi_3| - i|\psi_3\rangle\langle\psi_2| + |\psi_1\rangle\langle\psi_1| + |\psi_4\rangle\langle\psi_4|$. Write down the matrix corresponding to this operator. Is it Hermitian? Is it unitary? If it is Hermitian and/or unitary calculate its eigenvalues.

   (c) Is there any state $|\psi\rangle$ which obeys $A|\psi\rangle = 0$? If so, construct such a state.

   (d) Is it possible for $A \times E_0$, where E_0 is a scalar with units of energy to represent the Hamiltonian of some physical system? Why or why not?

2. Assume we have "crystal" made of only 3 identical atoms. Each atom has one orbital (no spin) with energy E_a.

   (a) If the atoms are arranged on a line such that an electron on an atom can only tunnel to its nearest-neighbor with a rate -$|A|/\hbar$ write down the 3×3 matrix Hamiltonian. Shift the overall energy to simplify the matrix. Either by hand, or using Mathematica, diagonalize the matrix to get the eigenvalues and eigenstates.

   (b) Now suppose the atoms are arranged on a ring so that the last site can tunnel to the first site. Write down the 3×3 Hamiltonian matrix. Is it the same or different than part (a)? Shift and then diagonalize this matrix to get the eigen-states and eigenvalues.

   (c) Only one of these two problems is translationally invariant. Which one? We know that if translation symmetry is preserved then the momentum p is a good quantum number, that is the operator $\hat{p}$ can be simultaneously diagonalized along with the Hamiltonian. Construct momentum eigenstates out of the Hamiltonian eigenstates. What are the corresponding momentum values p for each of the 3 simultaneous eigenstates?

3. Take the previous problem, but consider N atoms instead of 3. Take $E_a = 0$ and let the nearest neighbor tunneling rate be $-|A|/\hbar$. Assume periodic boundary conditions.

   (a) Write down the form of the N energy eigenvalues and N eigenstates for this system. **HINT: Look at the notes from class**

   (b) Suppose we start an electron in the orbital on site 1. Calculate the probability it stays on site 1 as a function of time.

4. Show that the kinetic energy of a three-dimensional gas of N free electrons at 0 K is

$$U_0 = \frac{3}{5}N\epsilon_f. \tag{1}$$

5. (a) Derive a relation connecting the pressure and volume of an electron gas at 0 K. Hint: Use the result of Problem 3 and the relation between ϵ_f and the electron concentration. The result may be written as $p = \frac{2}{3}(U_0/V)$.

   (b) Show that the bulk modulus $B = -V(\partial p/\partial V)$ of an electron gas at 0K is $B = 5p/3 = 10U_0/9V$.

   (c) Estimate the value of the bulk modulus B for the metal potassium which has a Fermi energy of $1.12eV$, and an electron density of 2.60×10^28 electrons per m^3.

6. For a free Fermi gas of electrons (with spin) derive the density of states $D(\epsilon)$ in 1d AND 2d. Hint: The result for 3d is proven in Chapter 4, and the answer for 2d is given in the next problem.

7. Show that the chemical potential of a Fermi gas in two dimensions is given by

$$\mu(T) = k_B T \ln\left[\exp\left(\pi n \hbar^2 / m k_B T\right) - 1\right] \tag{2}$$

for n electrons per unit area. Note: The density of orbitals of a free electron gas in two dimensions is independent of energy: $D(\epsilon) = m/\pi\hbar^2$, per unit area of specimen.

8. (a) For the drift velocity theory in a uniform magnetic field ($\mathbf{B} = B\hat{z}$) in Chapter 3, show that the static current density can be written in matrix form as

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \frac{\sigma_0}{1 + (\omega_c \tau)^2} \begin{pmatrix} 1 & -\omega_c \tau & 0 \\ \omega_c \tau & 1 & 0 \\ 0 & 0 & 1 + (\omega_c \tau)^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}. \tag{3}$$

where $\sigma_0 = ne^2\tau/m$ is the Drude conductivity at zero magnetic field and $\omega_c = eB/m$ is the cyclotron frequency at a magnetic field B.

(b) In the high magnetic field limit of $\omega_c \tau \gg 1$, show that

$$\sigma_{yx} = nec/B = -\sigma_{xy}. \tag{4}$$

The quantity σ_{xy} is called the Hall conductivity.

(c) Dropping all terms of order $(1/\omega_c \tau)^n$ for $n \geq 1$ determine σ_{xx} in this limit.