The last thing we discussed was interactions between local spins in materials, which I argued was captured by the "Heisenberg Hamiltonian".

\[
H = -\frac{1}{2} \sum_{\text{nearest neighbors}} J \mathbf{S}_i \cdot \mathbf{S}_j + g\mu_B \sum_{\text{all spins}} \mathbf{B} \cdot \mathbf{S}_i.
\]

The origin of these interactions is the exchange mechanism, and tells us that neighboring spins like to align (if \(J > 0\)) or anti-align (if \(J < 0\)).

This will clearly have consequences, for example, there is now an additional benefit for spins to align with field for \(J > 0\) and cost for \(J < 0\), affecting the balance between energy and entropy \(\rightarrow\) magnetization evolves as so.

\[K\]

\[
\begin{align*}
J > 0 \\
J < 0 \\
J = 0
\end{align*}
\]

And at \(T = 0\), spins will spontaneously choose directions \(\rightarrow\) "order".

\[
T = 0
\]

\[
\begin{array}{cccc}
\uparrow & \uparrow & \uparrow & \uparrow \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\end{array}
\]
TREATING THE FULL N-SPIN PROBLEM EXACTLY IS NOT A SIMPLE MATTER. IT TURNS OUT (AT LEAST MUCH HARDER THAN THE INDEPENDENT SPIN MODEL).

ONE APPROXIMATION WHICH IS OFTEN MADE TO GET A BASIC IDEA OF WHAT HAPPENS COMPRISSES A SERIES OF CALCULATIONS KNOWN AS "MEAN FIELD THEORY." HERE IS THE BASIC IDEA:

1. Focus on 1 spin
2. Assume the effect of all spins can be captured by the average (mean) field induced at the site of that spin
3. Assuming that all spins see the same average field
4. Enforce self-consistency
5. Solve

This strategy neglects a lot of stuff (e.g., quantum fluctuations), but usually reproduces general behaviors.
This is best seen through an example.

**The Ferromagnetic $S = \frac{1}{2}$ Ising Model**

Consider the above Hamiltonian applied to a bunch of $S = \frac{1}{2}$ spins. If $\mathbf{B} = B_0 \hat{z}$, there are only two spin possibilities for each site $S_z = \frac{1}{2}$ or $S_z = -\frac{1}{2}$ (the "Ising model").

The full Hamiltonian reduces to

$$
\mathcal{H} = -\frac{1}{2} \sum_{\langle i,j \rangle} J S_z^i S_z^j + g \mu_B B \sum_i S_z^i
$$

If we focus on a single site, $i$, that spin has

$$
\mathcal{H}_i = \left( g \mu_B B - J \sum_j S_z^j \right) S_z^i
$$

where the sum is over neighbors for given $S_z^j$. The $\left( \right)$ just looks like an effective ("molecular") field

$$
B_{\text{eff}}^i = B - \frac{J \sum_j S_z^j}{g \mu_B}
$$

In the main principle of mean field theory is that we can replace $B_{\text{eff}}^i$ with its average value ("mean field") and this value will, on average, be the same for all sites $B_{\text{eff}} \rightarrow \langle B_{\text{eff}} \rangle$. 
\[ H_i = g \mu_B \langle \text{Beff} \rangle S_i^z \]

which is just the lone Spin-\(\frac{1}{2}\) Hamiltonian considered last class using the same analysis.

\[ Z_i = e^{-\frac{g \mu_B \langle \text{Beff} \rangle}{k_B T}} + e^{-\frac{g \mu_B \langle \text{Beff} \rangle}{k_B T}} \]

\[ \langle S_i^z \rangle = -\frac{1}{2} \tanh \left( \frac{g \mu_B \langle \text{Beff} \rangle}{2 k_B T} \right) \]

But if we are just averaging, there is no reason that \( \langle S_j^z \rangle \) is different on any site, \( j \).

\[ \langle \text{Beff} \rangle = B - \frac{J}{g \mu_B} \sum_i \langle S_i^z \rangle \]

\[ = B - \frac{J}{g \mu_B} \sum_i \langle S^z \rangle \]

\[ = B - \frac{J}{g \mu_B} \cdot \frac{Z}{Z} \langle S^z \rangle \]

Where \( Z = \# \) nearest neighbors ("coordination number")
THE ONLY THING LEFT TO DO IS ENFORCE SELF-CONSISTENCY

\[ \langle S_z^2 \rangle = -\frac{1}{2} \tanh \left( \frac{g \mu_B B - Jz \langle S_z^2 \rangle}{2 k_B T} \right) \]

WHEN THIS EQUATION DOESN'T HAVE A CLOSED FORM SOLUTION, BUT CAN BE SOLVED NUMERICALLY, ONCE \( \langle S_z \rangle \) IS FOUND, IT CAN BE ASSOCIATED WITH A MEAN MOMENT \( \mu_s = -g \mu_B \langle S_z \rangle \) PER SITE AND NET MAGNETIZATION

\[ M = -gN \mu_B \langle S_z \rangle \]

SOLVING THE TRANSCENDENTAL EQUATION COMES DOWN TO FINDING THE INTERSECTION OF A STRAIGHT LINE (LHS OF EQ) WITH A HYPERBOLIC TANGENT (RHS OF EQ) THAT VARIES WITH TEMPERATURE, AS I STATED, IT CANNOT BE SOLVED ANALYTICALLY. BUT YOU CAN EASILY NOTE SOME GENERAL FEATURES

THEN

\[ \langle S_z \rangle = \pm \frac{1}{2} \tanh \left( \frac{Jz \langle S_z \rangle}{2 k_B T} \right) \]

WHERE I HAVE USED \( \tanh(-x) = -\tanh(x) \)
For high temperatures \( k_B T \gg J z \left\langle S_z \right\rangle \), the RHS of the EG is "flat" and the two sides plotted together look as so

Clearly this is only one interception at \( \left\langle S_z \right\rangle = 0 \), which is consistent with them being randomized at high \( T \) (go entropy!)

The situation changes as \( T \) gets smaller. The slope of the line becomes larger and larger until, below some critical temperature \( (T_c) \), you start encountering a situation that looks like

Now there are three solutions! \( \left\langle S_z \right\rangle \), which can be shown to be unstable

and \( \left\langle S_z \right\rangle = \pm \sigma(T) \)
Thus the system evolves with temperature from $\langle \sigma \rangle = 0$ to some finite value $\langle \sigma \rangle$.

Order!

Whether $\langle S_z \rangle = +\sigma$ or $-\sigma$ is chosen spontaneously (some fluctuation) or is dictated by some small environmental bias (a small field).

For ferromagnets, the critical temperature $T_c$ is often called the Curie temperature. This temperature is precisely when the two curves are tangential.

Using the expansion

$$\tanh \left( \frac{J z S_z}{2k_B T} \right) = \frac{J z \langle S_z \rangle}{2k_B T}$$

we can thus find $T_c$ by setting slopes of LHS and RHS equal.

$$1 = \left( \frac{J z}{2k_B T_c} \right)$$

$$k_B T_c = \frac{J z}{4}$$

Order happens suddenly at $T_c$ proportional to $J$. 
This was a $B=0$ result; you get a non-zero net magnetization at low temps spontaneously — the very definition of ferromagnet.

We can still explore the regime $T > T_c$ thought to find the general $\chi(T)$. Consider now the full self consistency equation (without setting $B=0$)

$$\langle S_z \rangle = -\frac{1}{2} \tanh \left( \frac{g \mu_B B - J_z \langle S_z \rangle}{2k_B T} \right)$$

At high $T$, there is no spontaneous $\langle S_z \rangle$ (we just saw) and any $\langle S_z \rangle$ is thus a response to $B$. For small $B$ then, $\langle S_z \rangle$ is small

$$\langle S_z \rangle \approx -\frac{1}{2} \tanh \left( \frac{g \mu_B B - J_z \langle S_z \rangle}{2k_B T} \right)$$

$$\approx -\frac{1}{2} \left( \frac{g \mu_B B - J_z \langle S_z \rangle}{2k_B T} \right)$$

$$4k_B T \langle S_z \rangle = J_z \langle S_z \rangle - g \mu_B B$$

$$\langle S_z \rangle = \frac{g \mu_B B}{J_z - 4k_B T} \left( T - T_c \right)$$
THEN \[ M = -n g \mu_B \langle S_z \rangle \]

\[ \chi = \frac{\mu_0 n g^2 \mu_B^2}{4 k_B (T - T_c)} \]

IF YOU RECALL

\[ \chi_{\text{Curie}} = \frac{\mu_0 n g^2 \mu_B^2}{3 k_0 T} \]

\[ = \frac{\mu_0 n g^2 \mu_B^2}{4 k_0 T} \quad \text{for } J = S = \frac{1}{2} \]

\[ \Rightarrow \chi_{\text{CW}} = \chi_{\text{Curie}} \left( \frac{1}{1 - \frac{T}{T_c}} \right) \]

"CURIÉ-WEISS LAW"

THE RESULT IS EXACTLY AS WE EXPECTED FROM BASIC ENERGY CONSIDERATIONS: MAGNETIZATION AT ALL TEMPS IS A BIT HIGHER DUE TO ENERGY GAIN FROM EXCHANGE.

IN FACT, WE SEE THAT \( \chi(T) \) DIVERGES AS \( T \to T_c \), WHICH MAKES SENSE, AS SPINS APPROACH TEMPERATURES WHERE THEY WANT TO ALIGN SPONTANEOUSLY, THEY WILL BE INFINITELY SUSCEPTIBLE TO SUGGESTIONS TO ALIGN BY APPLIED FIELDS.

\[ \chi_{\text{CW}} \]

\[ T_c \]

THIS IS A COMMON WAY OF MEASURING \( J \)
Extension to the AF (J < 0) case

Marginaly more complex is the case J < 0.
If, as we might guess, the preferred state looks like so

```
↑ ↓ ↑ ↓
↓ ↑ ↓ ↑
```

Then not all spins are equivalent. (We have broken discrete translational symmetry of the lattice.) Some spins will have opposite \(\langle S_z \rangle\) than others, and thus opposite \(\langle B_{eff} \rangle\).

We can get around this by considering "staggered" magnetization, where we separate the lattice into 2 sublattices, and assume the 2nd lattice is exactly opposite the 1st lattice

\[ M_{\text{staggered}} = -g \mu_B \left( \langle S_z \rangle_1 - \langle S_z \rangle_2 \right) \]

This gives the result

\[ \chi_{cw} = \frac{\chi_{\text{Curie}}}{1 + \frac{T_c}{T}} \]

You will do this on HW 6.