Newtonian Mechanics

Alexander Pope's famous verse runs

> "Nature, and Nature's laws, lay hid in night; God said, Let NEWTON be! and there was light."

and this is perhaps not an outrageous exaggeration; certainly, Newton's work marks the beginning of modern physics as we know it. Indeed, insofar as we can neglect the effects of relativity (which enter only for very high velocities) and of quantum mechanics (which generally speaking enter only on the atomic scale), we still believe that the description he gives of "the motion of bodies" is correct. Actually, Newton was a considerable polymath; within the area of physics he of course made fundamental contributions not only to mechanics (which is what I shall be concentrating on for the purposes of this course) but also to optics, and outside physics he gained a reputation in his lifetime as both an administrator and a theologian. He was, in fact, a fundamentally religious person, and almost certainly approved of the remark made in Cotes' preface to the 1713 edition of his *Principia*, to the effect that

> "He has so clearly laid open and set before our eyes the most beautiful frame of the System of the World, that if King Alphonso were now alive, he would not complain for want of the graces either of simplicity or of harmony in it. Therefore we may now more nearly behold the beauties of Nature, and entertain ourselves with the delightful contemplation; and, which is the best and most valuable fruit of philosophy, be thence incited the more profoundly to reverence and adore the great Maker and Lord of all. He must be blind who from the most wise and excellent contrivances of things cannot see the infinite Wisdom and Goodness of their Almighty Creator, and he must be mad and senseless who refuses to acknowledge them. Newton's distinguished work will be the safest protection against the attacks of atheists, and nowhere more surely than from this quiver can one draw forth missiles against the band of godless men."

What were Newton's achievements in the area of mechanics? In his classic book *Philosophiae Naturalis Principia Mathematica* ('The Mathematical Principles of Natural Philosophy') (1686),

1. He formulated the fundamental laws of (classical) dynamics which we now know as Newton's laws.

2. He developed (simultaneously with Leibniz in Germany) the branch of mathematics we now know as the differential calculus, which is essential to explore in detail the consequences of his laws.

3. He formulated the law of universal gravitation.

4. He unified (compare the story of the falling apple) the phenomena of terrestrial and celestial gravitation, and drew a host of consequences, particularly for planetary motion.

In this lecture I will review, more or less following Newton's own style of definitions and axioms (but in somewhat different order), the basics of Newtonian mechanics, and in the next lecture will explore the philosophical presuppositions and implications of his work.

Newton took from Galileo the important distinction between the *description* of motion ("kinematics") and the investigation of the *causes* of motion for any physical system ("dynamics"). Let's start with the description of motion:

A. Definitions ("kinematics")

- Absolute space:

> "Absolute space, in its own nature, without relation to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies; and which is commonly taken for immovable space; such is the dimension of a subterraneous, an aerial, or celestial space, determined by its position in respect of the earth. Absolute and relative space are the same in figure and magnitude; but they do not remain always numerically the same. For if the earth, for instance, moves, a space of our air, which relatively and in respect of the earth remains always the same, will at one time be one part of the absolute space into which the air passes; at another time it will be another part of the same, and so, absolutely understood, it will be continually changed."

- Absolute time:

> "Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external, and by another name is called duration: relative, apparent, and common time, is some sensible and external (whether accurate or unequable) measure of duration by the means of motion, which is commonly used instead of true time; such as an hour, a day, a month, a year."

- Absolute motion:

> "Absolute motion is the translation of a body from one absolute place into another; and relative motion, the translation from one relative place into another. Thus in a ship under sail, the relative place of a body is that part of the ship which the body possesses; or that part of the cavity which the body fills, and which therefore moves together with the ship: and relative rest is the continuance of the body in the same part of the

> ship, or of its cavity. But real, absolute rest, is the continuance of the body in the same part of that immovable space, in which the ship itself, its cavity, and all that it contains is moved. Wherefore, if the earth is really at rest, the body, which relatively rests in the ship, will really and absolutely move with the same velocity which the ship has on the earth. But if the earth also moves, the true and absolute motion of the body will arise, partly from the true motion of the earth, in immovable space, partly from the relative motion of the ship on the earth; and if the body moves also relatively in the ship, its true motion will arise, partly from the true motion of the earth, in immovable space, and partly from the relative motions as well of the ship on the earth, as of the body in the ship; and from these relative motions will arise the relative motion of the body on the earth. As if that part of the earth, where the ship is, was truly moved towards the east, with a velocity of 10010 parts; while the ship itself, with a fresh gale, and full sails, is carried towards the west, with a velocity expressed by 10 of those parts; but a sailor walks in the ship towards the east, with 1 part of the said velocity; then the sailor will be moved truly in immovable space towards the east, with a velocity of 10001 parts, and relatively on the earth towards the west, with a velocity of 9 of those parts."

Other concepts which are fundamental in Newton's work:*

(a) "Quantity of matter", i.e. mass; somewhat surprisingly, Newton defines it (*Principia*, p. 1) as the product of density and volume (from a modern point of view, it seems more natural to take mass as the primitive concept and define density as the ratio of mass to volume). Newton explicitly notes (p. 1) that the mass of a body "is proportional to the weight, as I have found by experiments on pendulums" and later (p. 304) remarks that "by experiments made with the greatest accuracy, I have always found the quantity of matter in bodies to be proportional to their weight", but does not seem to feel that this proportionality is particularly remarkable (it took Einstein, 230 years later, to bring out its significance).

(b) "Quantity of motion" (momentum, in modern terms) is defined as mass $\times$ velocity. Newton does not feel it necessary at this point to define velocity, but we need to, so a short digression:

Consider the motion of a body in a straight line, and mark off position ("coordinate") x along this line, with some arbitrary zero (more on this in the next lecture). The motion of the body can then be described by giving the value of time, t, as a function of x, that is, by giving the times at which different points on the line are reached. Actually, it is more convenient and natural† to give the information the other way around, i.e., to take

*References are to the Motte-Cajori translation of the *Principia* (University of California Press, 1966)

†Because, inter alia, the body can be at the same position at several different times, but for a given time can be at only one position.

time t as the "independent" variable and ask "At a given time t, what was the position x?" Imagine that we space the different values of t for which we ask the question more and more closely, so that in the end t becomes a "continuous" variable; then we are asking for x as a function of t. We have several ways of representing the answer, e.g.:

1. give a table of numbers, that is, the numerical value of x for each (closely spaced) value of t.

2. draw a two-dimensional graph, in which t is represented by the horizontal axis and x by the vertical one.

3. give an explicit algebraic formula: e.g. for free motion $x = \text{const} \times t$, or for free fall under gravity, $x = at^2/2$

It is a *fact of observation* that under normal conditions x is a relatively "smooth" function of t.

We are now in a position to define the concept of velocity. First, we pick two times which are "close" together, t_1 and t_2 and define a sort of "average" velocity by[‡]

$$v_{\text{av}} \equiv \frac{x_2 - x_1}{t_2 - t_1}$$

The quantity v_{av} clearly depends in general on how far apart the times t_1 and t_2 are. However, if we make them closer and closer together, then provided the curve of x against t is "smooth" the expression for v_{av} obtained in this limit is just the slope of the curve at time t and is well-defined. So we can define the true velocity v at time t by

$$v \equiv \lim_{t_2 \to t_1} \frac{x_2 - x_1}{t_2 - t_1} \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

(equivalent to $\frac{dx}{dt}$ in the standard notation of the differential calculus). Note that v is in general a function of t.

Although the idea of an "instantaneous velocity" which may depend on time is nowadays a basic concept in physics, it may be heartening for those who find it difficult to grasp that Galileo's contemporaries evidently had equal difficulty; see his *Dialogue on the Two New Sciences*, p. 164.

One other notion we need is that of "composition of velocities". Taking the concept of "mutually perpendicular" (or "right angle") as intuitively given, let's use equally spaced horizontal and vertical lines to mark off a two-dimensional "grid" which will define two different coordinates, say x (horizontal) and y (vertical). (Unfortunately, we can now no longer represent time by the horizontal axis and have to just represent it by an algebraic symbol). We can now discuss the quantities x and y separately as a function of t: $x = x(t)$, $y = y(t)$, and define the two different corresponding "components" of velocity v_x , v_y by

$$v_x \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \ \left(\equiv \frac{dx}{dt}\right), \quad v_y \equiv \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} \ \left(\equiv \frac{dy}{dt}\right)$$

[‡]The symbol "$\equiv$" is conventionally used in mathematics and physics to denote a definition.

It is natural also to define the *total* velocity v (or "speed") as the ratio of total distance traveled (call it Δs) to the time Δt elapsed. But by Pythagoras' theorem, $(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$, and hence if we define as suggested

$$v \equiv \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

then we have

$$v^2 = v_x^2 + v_y^2$$

The great advantage of this procedure[§] (which we can go through for any quantity having a "direction" associated with it) is that it allows us to discuss the different "components" v_x , v_y of velocity separately. Note that the choice of the axes x and y is arbitrary, provided that the "grid" so generated is indeed square (otherwise Pythagoras' theorem does not apply). I have gone through the procedure explicitly for two dimensions, but it is easy to see how to generalize it to three; in this case the conventional choice of axes is to have two horizontal and one vertical. I'll return to this question in Lecture 7.

Returning now to Newton's definitions, we also have

(c) "The *vis insita*, or innate force of matter, is a power of resisting, by which every body, as much as in it lies, continues in its present state, whether it be of rest, or of moving uniformly forwards in a right line . . . "

"Resistance is usually ascribed to bodies at rest, and impulse to those in motion; but motion and rest, as commonly conceived, are only relatively distinguished; nor are those bodies always truly at rest, which commonly are taken to be so".

(d) Force: An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of uniform motion in a right line. This force consists in the action only, and remains no longer in the body when the action is over. For a body maintains every new state it acquires, by its inertia only. But impressed forces are of different origins, as from percussion, from pressure, from centripetal force."

In some sense these passages are the most definitive statement of the final break with Aristotelian physics. It is now no longer rest, but rather uniform motion which is the "natural" state and needs no explanation – a force is needed only to set a body, initially at rest, in motion. Thereafter the body maintains the new state of motion it has acquired "by its inertia only". ("*Vis insita*" seems to be used by Newton more or less interchangeably with "inertia", which has essentially its modern sense).

It is interesting that Newton here seems to be conscious of the danger of circularity and so goes out of his way to specify examples of physical forces. Note also that it is implicit, here, that force has a direction (and hence, like velocity, can be "resolved" into "components").

[§]It is sometimes called "resolving" the velocity into its "components".

B. Newton's laws of motion

1st law "Projectiles continue in their motions, so far as they are not retarded by the resistance of the air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are continually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by the air. The greater bodies of the planets and comets, meeting with less resistance in freer spaces, preserve their motions both progressive and circular for a much longer time".

In modern notation, this states: the velocity of a body is constant in time if there are no external forces acting on it.

2nd law "The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed". In modem notation: The rate of change of momentum is equal to the applied force, both in magnitude and in direction. Note that since momentum is defined as mass × velocity (see (b) above) this implies the notion of a second time derivative: if mass = constant, then rate of change of momentum = mass × rate of change of velocity ≡ mass × acceleration. What exactly do we mean by this? We can define acceleration a by

$$a \equiv \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} \equiv \lim_{t_2 \to t_1} \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

However, the velocity v is itself defined as a derivative, $v \equiv \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$, and hence we have

$$a \equiv \lim_{\Delta t \to 0} \frac{\Delta \left(\Delta x / \Delta t \right)}{\Delta t}$$

($\equiv \frac{d^2x}{dt^2}$ in the standard notation of the differential calculus). In graphical terms, a is the curvature (rate of change of slope) of the graph of x versus t. I return in lecture 7 to the question of whether the second law is "circular", i.e. merely a definition of "force".

3rd law "To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts". "...If a body impinge upon another, and by its force change the motion of the other, that body also (because of the equality of the mutual pressure) will undergo an equal change, in its own motion, towards the contrary part. The changes made by these actions are equal, not in the velocities but in the motions of bodies; that is to say, if the bodies are not hindered by any other impediments. For, because the motions are equally changed, the changes of the velocities made towards contrary parts are inversely proportional to the bodies. This law takes place also in attractions, as will be proved in the next Scholium". In modern notation: If two bodies, 1 and 2 interact, then the change of momentum of 1 is equal and opposite to the change of momentum of 2, and hence the velocity changes v_1 and v_2 satisfy the relation

$$\Delta v_1 / \Delta v_2 = -m_2/m_1$$

– a result which will be familiar, at least qualitatively, to anyone who has ever tried to step out of a punt which is not moored to the bank! An alternative formulation is: The total momentum of a system of bodies on which no external forces act is conserved (i.e. is constant in time).

Note that each of Newton's laws applies to each of the different "components" of velocity, etc., separately. E.g. suppose that a body moves in the Earth's gravitational field. Then there is no force in the horizontal direction, so by the first law the horizontal components of momentum are constant, and since the mass is constant, this means that the components of velocity are also constant. On the other hand, in the vertical direction there is a force (gravity); thus, by the second law the rate of change of vertical velocity (vertical acceleration) is the force divided by the mass, which in this case is just the well-known gravitational acceleration g.

C. The law of universal gravitation

(Principia, book III, Prop. VII and Cor. II)

"That there is a power of gravity pertaining to all bodies, proportional to the several quantities of matter which they contain": "the force of gravity towards the several equal particles of any body is inversely as the square of the distance from the particles".

In modern terminology: Between any two bodies of mass m_1, m_2 there is a gravitational force of attraction, directed along the line between them[¶] and proportional to the inverse square of the distance. In symbols,

$$F = G\,\frac{m_1 m_2}{r^2}$$

where r is the distance between the bodies. (G is the so-called Cavendish or gravitational constant). The really crucial insight, here, is that *the same* force as draws the mythical apple to the ground, sustains the motion of the planets in their orbits. A telling passage here is the following one (*Principia*, p. 3):

> "...If a leaden ball, projected from the top of a mountain by the force of gunpowder, with a given velocity, and in a direction parallel to the horizon, is carried in a curved line to the distance of two miles before it falls to the ground; the same, if the resistance of the air were taken away, with a double or decuple velocity, would fly twice or ten times as far. And by increasing the velocity, we may at pleasure increase the distance to which it might be projected, and diminish the curvature of the line which it might describe, till at last it should fall at the distance of 10, 30, or 90 degrees, or even might go quite round the whole earth before it falls; or lastly, so that it might never fall to the earth, but go forwards into the celestial spaces, and proceed in its motion *in infinitum.* And after the same manner that a projectile, by the force of gravity, may be made to revolve in an orbit, and go round the

[¶]Or (for spherical bodies) between their centers. Such forces are often called "central".

> whole earth, the moon also, either by the force of gravity, if it is endued with gravity, or by any other force, that impels it towards the earth, may be continually drawn aside towards the earth, out of the rectilinear way which by its innate force it would pursue; and would be made to revolve in the orbit which it now describes; nor could the moon without some such force be retained in its orbit."

Although Newton, characteristically, does not tell us how he hit upon the law of universal gravitation, and in particular the inverse-square aspect, it seems likely that having hit on the general idea he deduced the details from Kepler's empirical laws of planetary motion. Recall that these are:

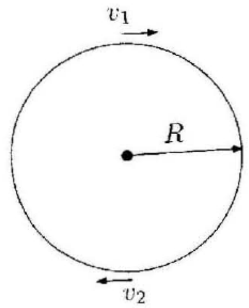
1. The planets move in elliptical orbits with the Sun at one focus.
2. Equal areas are swept out in equal times.
3. The square of the orbit period is proportional to the cube of the (suitably averaged) distance from the Sun.

How do Newton's laws of motion, plus his law of universal gravitation, explain these empirical facts?

1. While the elliptical shape of the orbit requires a detailed calculation (and turns out to depend crucially on the law being exactly inverse-square), it is fairly easy to see that Newton's first law plus the "central" nature of the gravitational force can explain at least the fact that the plane of the orbit contains the Sun. At any instant the velocity of the planet, and the line connecting it to the Sun, together define a plane: let us choose our coordinate system so that it is the xy plane, Now, the gravitational force is directed towards the Sun and thus lies in the xy plane; there is no force, and hence no acceleration, in the z direction (i.e, the direction perpendicular to the plane). But acceleration is just rate of change of velocity, and thus since originally, by construction, the velocity had no z component, it never acquires any. Thus the orbit remains forever in the xy plane as required. Note that this argument is independent of the distance dependence of the gravitational force.

2. Kepler's second law is automatically satisfied for the case of zero external force (i.e. uniform motion in a straight line) since the area of a triangle is half its base times its perpendicular height, and the latter is constant (cf. Feynman, p. 36). For the proof in the case of a "central" force such as gravitation, see the original argument of Newton as reproduced in Feynman, pp. 35-7 (but note, as Feynman does not do explicitly, that the line between the points he marks 3 and 4 is parallel to the radius (i.e., the line connecting the planet to the Sun). Again, the result depends only on the "central" property of the force and is independent of the inverse-square aspect.

3. The proof of Kepler's third law is a bit more tricky and I shall give it only for circular orbits (actually a pretty good approximation for the Earth and the inner planets). We assume that since the mass of the Sun is huge compared to that of any of the planets, we can neglect the recoil of the Sun, i.e. regard the latter as a fixed "center of force".



Consider the application of Newton's second law to the horizontal component of the velocity as shown on the diagram. Suppose we average it over half a period, then we have

$$\text{average horizontal acceleration } \bar{a} = \text{ average gravitational force/mass}$$

Now the speed of the planet in its orbit is constant and equal to the orbit circumference, $2\pi R$, divided by the period T; since the horizontal component is reversed in half a period we have

$$v_2 = -v_1 = 2\pi R/T, \quad \bar{a} = \Delta v/\Delta t = (v_1 - v_2)/(T/2) = 8\pi R/T^2$$

As to the average horizontal component of gravitational force/mass, this is GM_S/R^2 (G = gravitational constant, M_S = mass of Sun) $\times$ the average horizontal fraction of the line joining the planet to the Sun over the half period indicated. If we call this fraction $c^\dagger$

$$\frac{8\pi R}{T^2} = c \cdot \frac{GM_\mathrm{S}}{R^2}$$

or

$$T^2 = \frac{8\pi}{c} \cdot \frac{R^3}{GM_\mathrm{S}} = \mathrm{const} \cdot R^3 \qquad (*)$$

as stated by Kepler's third law. Note that if we know the absolute distances of the planets from the Sun (and know the value of c), we can infer from their periods the product GM_S(but not M_S by itself).

Let's finally apply this argument, as Newton did, to the motion of the Moon around the Earth. According to the law of universal gravitation, the gravitational acceleration due to the Earth at the position of the Moon is independent of the mass of the latter

† It is actually $2/\pi$, but we do not need to know this to draw the conclusion $(*)$.

and a factor $r_{\mathrm{E}}^2/R_{\mathrm{M}}^2$ times that at the surface of the Earth (r_{E} = radius of Earth, R_{M} = distance of Moon from center of Earth). That is,

$$a_{\mathrm{Moon}} = g \cdot r_{\mathrm{E}}^2/R_{\mathrm{M}}^2$$

where g is gravitational acceleration at Earth's surface. Since** g is about $10\,\mathrm{m/sec}^2$, $r_{\mathrm{E}} \approx 6400\,\mathrm{km}$ and R_{M} (as measured e.g. by triangulation) $\approx 3.8\times10^5$ km, the acceleration of the Moon in its orbit is about $2.8\,\mathrm{mm/sec}^2$. In which direction is this acceleration (call it a)? It has to be in the direction of the force, i.e. towards the Earth, and its magnitude is constant. Its average horizontal component $\bar{a}$ over the half-period considered above is $c \times a$ where $c(= 2/\pi)$ is the constant mentioned above, so using the result $\bar{a} = 8\pi R/T^2$ (see above) we find

$$a = 4\pi^2 R/T^2$$

(this is a standard result for a circular orbit). Thus for the moon we predict

$$T_M = 2\pi\sqrt{R_M/a}$$

If we put in the value of $a, \approx 2.8mm/sec^2$, derived above, we find that T_M is about $2.5 \times 10^6 secs$, i.e. just about 28 days! This was probably the first and most impressive pay-off of the "falling-apple" argument. Note, again, that we don't need to know the mass of the Moon to derive it; this is a quite general characteristic of bodies accelerated by the gravitational field of a much larger body such as the Earth – as Galileo observed, in the Earth's field all bodies accelerate at the same rate, independent of their mass, and this is true whether they are in Pisa or at the position of the Moon (though the acceleration is of course very different in the two cases).

** $\approx$ is the mathematical symbol for "is approximately equal to".