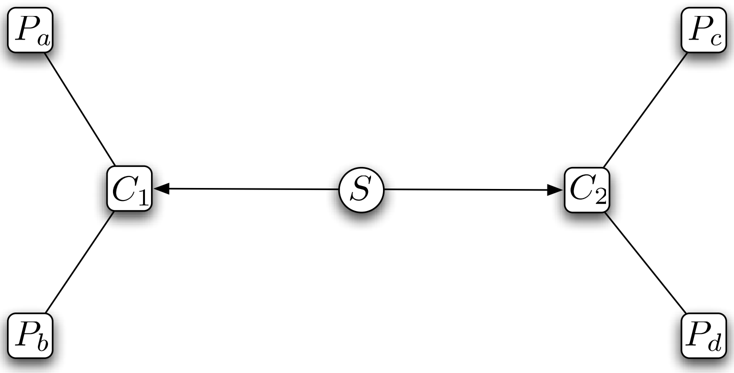
The Einstein-Podolsky-Rosen Thought Experiment and Bell's Theorem

As first shown by Bell (1964), the force of the arguments against HV theories is increased enormously when we consider not single beams of photons, but rather the *correlations* between the two photons emitted in a single atomic process. It is probably this system, more than any other, that best illustrates the fundamental weirdness of QM. Indeed, Bell's theorem has been described, in cold print and without irony, as "the most profound result of science."

Let's first describe the experimental setup. We consider a process in which an ensemble of atoms S is excited, e.g., by a spark discharge; thereafter, the atoms decay from the excited state to the groundstate by a two-step process involving an intermediate state and the emission of *two* photons. We will concentrate on cases in which the two photons are emitted "back-to-back", and moreover assume that we are confident that when we observe a pair of photons, that they have been emitted by the same atom[‡] and therefore may be appropriately correlated.



What we are going to do is to measure the polarization of each of the photons along various alternative axes; for reasons that will become clear, it is important that the system not "know" ahead of time which axis we are going to measure along. Since it is not feasible to rotate the polarizer on the necessary timescale, in practice this is achieved by switches C_1 and C_2, such that C_1 switches photon 1 into either a device (P_a) that "measures" the polarization along axis a, or a similar device (P_b) that "measures" it along a different axis b; similarly, C_2 switches photon 2 into either P_c or P_d. Note that the outcome of a "measurement" can be defined purely operationally; e.g., if the photon 1 in question registers a count in the detector D_a, placed behind polarizer P_a then *by definition* this photon is measured to have a polarization along a, while if no count is registered it is measured to have polarization perpendicular to a.[§] In the experiments of

[‡]Any cases in which we mistakenly assume a pair of photons to have been emitted by the same atom, when in fact they came from different ones, will tend to spoil the spectacular correlations predicted by QM; if the latter are found, this is, therefore, evidence that we have not made too many such mistakes.

[§]In the most sophisticated experiments of this type, the "rejected" photons (i.e., in this case, those

Alain Aspect and coworkers, the switches C_1 and C_2 are activated by (what is hopefully a good approximation to) a "random" process, in such a way that there is no time for the information about whether it was the a- or b-polarization that was measured of photon 1 to reach the far end of the laboratory in time to influence the outcome of the measurement made of photon 2 (the two events are spacelike-separated in the sense of special relativity).

To describe the output of this experiment, i.e., the basic experimental data that will be the subject of prediction and analysis, it is convenient to introduce four quantities A, B, C, D, which can each take only the values ± 1. Every time the photon 1 of a pair is switched into counter P_a we assign to it a value of A as follows: If the photon passes the polarizer (i.e., the counter D registers), then A is defined to be $+1$, while if it fails to pass it (no count) A is defined to be -1. Similar definitions are given for the quantities B, C, D. Note carefully that for any one pair of photons we can define *either* A *or* B, but not both: if the photon 1 is switched into counter P_a, then B is not defined, and vice versa. Similarly for C and D. However, there is nothing to stop us defining A and C simultaneously for at least some pairs, namely those of which photon 1 was switched into counter P_a and 2 into P_c. We can define, therefore, a *measured average* of the correlation as follows:

$$\langle AC \rangle \equiv \frac{1}{N_{ac}} \sum_i^{(ac)} A_i C_i$$

where the restriction (ac) in the sum over i means that we sum over all and only these pairs of photons of which photon 1 was switched into the measuring device P_a and photon 2 into the measuring device P_c (we have to make this restriction, since for all other pairs of photons at least one of the quantities A_i and C_i is not defined!); N_{ac} is the total number of such pairs. In a similar way, we can define the correlations $\langle AD \rangle$, $\langle BC \rangle$ and $\langle BD \rangle$. It is the business of QM, and of various possible hidden-variable theories, to predict the values of these measurable correlations.¶ Note that

- strictly speaking, these averages are defined on different subensembles of the original ensemble of pairs produced in the atomic cascade process;

- for the reason given above, we cannot meaningfully define a "measured correlation" $\langle AB \rangle$ or $\langle CD \rangle$ (the quantities A_i and B_i are never simultaneously defined).

We have so far swept one crucial question under the rug: we have talked about "measured values" A_i, etc., but exactly where and when does the "measurement" take place? Almost all discussions of Bell's theorem and related matters in the literature assume, either explicitly or implicitly, that it takes place either when the photon in question "passes" or "fails to pass" the relevant polarizer, or at latest when it is "registered" in

reflected by the various polarizers) are directly counted. This gets around the difficulty that if no count is registered, it might be because there was no photon there in the first place!

¶And, of course, of other measurable averages such as $\langle A \rangle$ – which, however, are less directly relevant in the present context.

the cathode of the photomultiplier (detector). We shall see later (lecture 21) that this view begs a fundamental question, but let us, for the purposes of the analysis in this lecture, make this conventional assumption.

First, let's discuss the QM description of the experiment and the predictions that it makes for the correlations.[‖] We are interested in (e.g.) the probability that when counter 1 is set in direction a and counter 2 in direction c, photon 1 passes and photon 2 is rejected. In QM, this probability has, according to our general conceptions, to be represented as the square of a probability amplitude, and we therefore need to generalize this idea to apply to two coupled systems. Recall that in the case of a single photon beam, the states could always be specified by choosing an arbitrary direction – call it x – in the plane perpendicular to the direction of motion, and the direction y perpendicular to it in the plane: the state vector corresponding to a photon that is guaranteed to pass a polarizer set in the x-direction is then denoted $|x\rangle$ or $|\uparrow\rangle$, and that corresponding to certain rejection $|y\rangle$ or $|\rightarrow\rangle$. Then the arbitrarily polarized beam is described by a so-called "linear combination":

$$|e\rangle = e_x|x\rangle + e_y|y\rangle \qquad (\text{or } e_x|\uparrow\rangle + e_y|\rightarrow\rangle)$$

and the probability of passing a polarizer set in the $|\uparrow\rangle$ is e_x^2, etc. (The coefficients e_x and e_y are the probability amplitudes for passage or rejection; see appendix to lecture 19.)

In the case of two beams, we can again choose an arbitrary axis x in the plane perpendicular to the direction of propagation; until further notice it is convenient to choose it to be the same for the two beams. We then define four independent state vectors:

$$|\uparrow_1\uparrow_2\rangle, \quad |\uparrow_1\rightarrow_2\rangle, \quad |\rightarrow_1\uparrow_2\rangle, \quad |\rightarrow_1\rightarrow_2\rangle$$
$$(\text{or } |\mathrm{x_1x_2}\rangle, \quad |\mathrm{x_1y_2}\rangle, \quad |\mathrm{y_1x_2}\rangle, \quad |\mathrm{y_1y_2}\rangle)$$

The first, $|\uparrow_1\uparrow_2\rangle$, corresponds to a state in which both photons are guaranteed to pass a polarizer set in the x-direction. Similarly, $|\uparrow_1\rightarrow_2\rangle$ corresponds to the case in which the photon 1 will definitely pass a polarizer set in the x-direction while photon 2 will fail to pass a similarly oriented one: and so on. A more general state of polarization of the two beams is given by adding these state vectors with arbitrary amplitudes:

$$c_{\uparrow\uparrow}|\uparrow_1\uparrow_2\rangle + c_{\uparrow\rightarrow}|\uparrow_1\rightarrow_2\rangle + c_{\rightarrow\uparrow}|\rightarrow_1\uparrow_2\rangle + c_{\rightarrow\rightarrow}|\rightarrow_1\rightarrow_2\rangle \tag{$*$}$$

Then, e.g., the probability of both photons passing a polarizer set in the $\uparrow$ direction is $c_{\uparrow\uparrow}^2$. It should be noted that the idea of "adding" probability amplitudes used here goes very considerably beyond the simple and almost intuitive notion of adding amplitudes for a *single* beam. In that case, the operation of adding amplitudes had a close classical correspondence to the idea of adding different components of the electric field to get the total field. In the two-beam case, by contrast, there is in general no classical state that in any way corresponds to a state of the form ($*$).

[‖] At this point, you are encouraged to review the appendix to lecture 19.

For definiteness, let's consider a particular class of atomic transition processes, which is in fact widely used in the relevant experiments. The pairs of photons emitted in this kind of transition have the following remarkable property: we may choose any direction in the xy-plane to be our x-axis and set both polarizers to transmit photons polarized along this axis. Then, either both photons will be transmitted or both will be rejected, the probability of each outcome being 50% (This is an experimental fact that can be directly checked.) What does this imply about the coefficients in equation (∗)? Since, as we have seen, $|\uparrow_1\rightarrow_2\rangle$ describes a state in which 1 is transmitted and 2 is rejected, and we know from experiment that this never happens, we must exclude this amplitude, i.e., set $c_{\uparrow\rightarrow} = 0$. Similarly for $|\rightarrow_1\uparrow_2\rangle$. Thus, we are left with the expression $c_{\uparrow\uparrow}|\uparrow\uparrow\rangle + c_{\rightarrow\rightarrow}|\rightarrow\rightarrow\rangle$. Now, as we have seen, the probability of both photons passing the $|\uparrow\rangle$ polarizer is $c^2_{\uparrow\uparrow}$, and the probability of both being rejected is $c^2_{\rightarrow\rightarrow}$. Since we know from experiment that each of those two probabilities is 50%, we must have $c^2_{\uparrow\uparrow} = c^2_{\rightarrow\rightarrow} = \frac{1}{2}$. But this still allows four possibilities:** $c_{\uparrow\uparrow} = \pm\frac{1}{\sqrt{2}}$, $c_{\rightarrow\rightarrow} = \pm\frac{1}{\sqrt{2}}$. Let's suppose for the sake of argument that $c_{\uparrow\uparrow} = +\frac{1}{\sqrt{2}}$; are we then allowed to choose $c_{\rightarrow\rightarrow}$ to be either $+\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$, or perhaps to be + for some pairs and − for others? Certainly, if we only examine the transmission or rejection for x-setting of both polarizers, any of these possibilities would seem to predict the experimentally observed results.

However, it is crucial to remember that we need to guarantee "perfect correlation" (i.e., either both photons pass, or both fail) for *any* setting of the polarizers, provided only that it is the same for both photons. When this is taken into account, it turns out (Appendix) that if $c_{\uparrow\uparrow} = +\frac{1}{\sqrt{2}}$, then $c_{\rightarrow\rightarrow}$ must also be $+\frac{1}{\sqrt{2}}$. Similarly, if $c_{\uparrow\uparrow} = -\frac{1}{\sqrt{2}}$, then $c_{\rightarrow\rightarrow}$ is also $-\frac{1}{\sqrt{2}}$. Actually, it is not necessary to distinguish these two possibilities, since they give the same predictions for all possible experiments. Thus, we arbitrarily take $c_{\uparrow\uparrow} = +\frac{1}{\sqrt{2}}$, and the most general form of the two-photon state is uniquely determined to be

$$\psi(1,2) = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle + |\rightarrow\rightarrow\rangle\right) \tag{†}$$

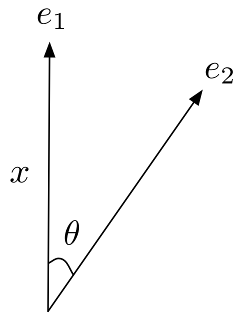
$$\left(\text{or } \frac{1}{\sqrt{2}} \quad (|\mathrm{x}_1\mathrm{x}_2\rangle \quad + \quad |\mathrm{y}_1\mathrm{y}_2\rangle)\right)$$

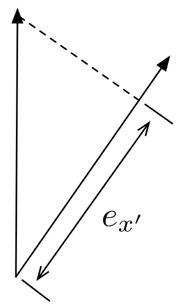
Notice that in deriving this result, we did not specify how we had chosen our x axis (↑-axis). Thus, this description must be true for an arbitrary choice of x axis! Indeed, this turns out to be true (cf. Appendix).

Once we know that the quantum state of the 2-photon system is the above one, it is straightforward to predict the correlations for *arbitrary* relative orientation of the two polarizers. Let us arbitrarily choose the transmission direction of polarizer number 1 to be the x (↑) axis, and let the component of x along the transmission direction of the second polarizer (x') be $e_{x'}$ $(= \cos\theta)$. (See figure.)

Suppose now that we do the experiment with a given pair of photons from the ensemble described by the state (†) and find that photon 1 has passed its polarizer. At

**Actually, the coefficients $c_{\uparrow\uparrow}$, etc., are complex numbers in the mathematical sense, so many more possibilities are allowed.





this point our information has changed, and we must now remove the component $| \rightarrow\rightarrow\rangle$, which corresponded (inter alia) to photon 1 failing. Thus, the pair must be ascribed to a new ensemble described simply by $| \uparrow\uparrow\rangle$; i.e., both 1 and 2 are polarized in the x-direction. But if photon 2 is polarized in the x-direction, then by (the quantum version of) Malus's law, the probability of its passing its own polarizer, which is set in the x'-direction, is simply $e^2_{x'}$ $(= \cos^2\theta)$, while the probability of rejection is $1 - e^2_{x'}$ $(= \sin^2\theta)$. Suppose now that photon 1 were rejected; then the new ensemble is $| \rightarrow\rightarrow\rangle$, so that photon 2 is now with certainty polarized along the y-axis. Now the probability of it passing its own polarizer is $e^2_{y'}$, which is simply $1 - e^2_{x'}$, or $\sin^2\theta$, and the probability of rejection is $e^2_{x'} = \cos^2\theta$. Thus we can summarize the results as follows:

If 1st passes:	*If 1st fails:*
$p(\text{2nd passes}) = \cos^2\theta$	$p(\text{2nd passes}) = \sin^2\theta$
$p(\text{2nd fails}) = \sin^2\theta$	$p(\text{2nd fails}) = \cos^2\theta$

Note, also, that the chance of 1 passing (with no information on the fate of 2) is 50%. Denoting "1 passes, 2 fails" by $(+,-)$, etc., we therefore have

$$p(+,+) = p(-,-) = \frac{1}{2}\cos^2\theta$$

$$p(+,-) = p(-,+) = \frac{1}{2}\sin^2\theta$$

Finally, we write down the result for the quantity $\langle AC\rangle$ defined above. To do so, we note that the product AC by definition $= 1$ for the $(++)$ and $(--)$ outcomes, and $= -1$ for $(+-)$ and $(-+)$. Using the trigonometric identity $\cos^2\theta - \sin^2\theta = \cos 2\theta$, we therefore finally get for the correlation in the two-photon quantum state

$$\langle AC\rangle = \cos 2\theta_{ac}$$

where θ_{ac} the angle between the polarizer settings a and c. Similarly, we would of course get, e.g.,

$$\langle BD\rangle = \cos 2\theta_{bd}$$

Note in particular that for identical polarizer settings, $\langle AC \rangle = +1$ (perfect correlation), as we have already seen. At first sight, there seems nothing particularly interesting about this result.

The EPR paradox

As a first step to seeing how "weird" the predictions of QM really are, let's review an argument put forward by Einstein, Podolsky and Rosen in a famous paper in 1935. Interestingly, the point of this argument was not to show that the experimental *predictions* of QM were wrong, or even were in conflict with apparently "common-sense" assumptions, but rather that the QM description of reality cannot be *complete*. The "EPR argument" was, historically, formulated in terms of position and momentum, but for our purposes, it is much more convenient to transcribe it to the case of measurements of polarization.

Let's recall that for a single photon beam we have the choice between, e.g., putting it through an "x-polarizer" ("$\uparrow$ or $\rightarrow$") or through an "x'-polarizer" ("$\nwarrow$ or $\nearrow$"). We know experimentally that a beam that is guaranteed to pass the x-polarizer will not necessarily pass an x'-polarizer (in fact it has a 50% chance of doing so). According to the standard (at least in 1935) Copenhagen interpretation of QM, this is an irreducible aspect of the QM description; it is not a question of missing information, but of an irreducible indeterminacy or complementarity. Thus, if the QM description of reality is complete, it is impossible for a system to possess simultaneously a definite value of the answer to the question "$\uparrow$ or $\rightarrow$?" and a definite value of the answer to "$\nwarrow$ or $\nearrow$?". EPR announce that they will refute this latter statement by showing that under certain conditions, definite answers to both these questions do simultaneously exist (in their language, both the x- and x'-polarizations have "simultaneous reality"), from which it will then follow that the QM description of reality cannot be complete.

To develop their argument, they start from the premise: "Every element of physical reality must have a counterpart in a complete theory." They then give what they regard as a sufficient (though possibly not necessary) criterion for what is to constitute an "element of reality". This criterion is crucial to their argument; it reads:

> If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

In other words, if at any time t we can predict with certainty the result q_i of a measurement of a quantity Q at time t, then there must exist (at time t) an element of physical reality corresponding to q_i. This seems, at first sight, eminently reasonable.

Now, continue EPR, consider a case in which, at some time t_0, a particular excited atom decays, producing a pair of photons that are in the so-called "0^+ state", which QM describes by the correlated quantum state (†). It is assumed that the experimental predictions of QM are in fact correct for this pair. Let us then wait until some time $t_1 > t_0$ when the photons in question are "far" apart, and at that time measure (say) the x-polarization of photon 2. If it passes the polarizer, we immediately know that photon 1 will pass a similarly set polarizer, and if 2 fails to pass, then 1 will also fail (perfect correlation: as already emphasized, we assume that the experimental prediction $[\langle AC\rangle = \cos 2\theta_{ac}$, and hence $\langle AC\rangle = 1$ for $a = c]$ is in fact correct). Thus, at time t_1 there exists, according to the EPR criterion, an "element of reality" for photon 1 corresponding to the answer to the question "$\uparrow$ or $\rightarrow$?". Moreover, since we have achieved the necessary ability to predict it without in any way disturbing the "system" (photon 1), this value must have existed also *before* the measurement, and indeed even if we had made no measurement at all! (Although, of course, in the latter case, we would never know what it was.)

But now it is clear that we could have gone through exactly the same argument regarding the x'-polarization: since by observing the result of an x' measurement on photon 2, we can with certainty predict its value for photon 1, it follows that there was already, before or even in the absence of the measurement, an "element of reality" corresponding to the answer to the question "$\nwarrow$ or $\nearrow$?". Thus, the conclusion of the argument is that in this situation, there exist simultaneously definite answers to the questions "$\uparrow$ or $\rightarrow$?" and "$\nwarrow$ or $\nearrow$?" – exactly what the QM description does not allow!

Let's emphasize again that the EPR argument does not itself show that the QM results are in any way "unreasonable" or even particularly odd. In fact, EPR seem to have envisaged that it would some day be possible to provide a "hidden-variable" underpinning to QM which would solve the dilemma without violating any particular "common sense" assumption. Actually, for the problem considered by EPR this expectation is quite correct; in fact, it is quite easy to construct a simple hidden-variable theory that will give perfect correlation for any choice of angle which is the same for both photons. We need only suppose that each pair of emitted photons is described at random by an "arrow" that may be in one of two perpendicular directions in the xy-plane, with probability 50% each (e.g., we could choose the arrows as $\nwarrow$ or $\nearrow$); and that when incident on a polarizer, a photon will pass if its arrow is within 45° of the polarizer axis (positive or negative) and otherwise be rejected. It is immediately clear that if the two polarizers are set in the same direction, whether both photons will pass or neither is 50% probability each – exactly the QM prediction. So… does that mean that a simple HV theory of this type, or some modification of it, can reproduce all the QM predictions? This question was left in the air – with many people probably vaguely suspecting the answer yes, though no such theory was ever explicitly constructed – until the work of Bell in 1964.

Bell's theorem

Although in Bell's original 1964 paper it was stated in a more restricted form, the usual statement nowadays of Bell's theorem is:

> No objective local theory can reproduce the experimental predictions of QM.

We need, of course, to define "objective local theory". Various possible definitions exist that for most practical purposes are effectively equivalent; for our purposes let's define an "objective local theory" by the conjunction of three properties:

1. Objectivity: An isolated system possesses a definite (though, of course, in general unknown) value of any quantity that may be measured on it.

2. Local causality: The value of a property possessed by an isolated system cannot be affected by any operations carried out at sufficient "separation" from it. (A more rigorous version specifies "sufficient separation" as equivalent to the statement that the relevant events are spacelike-separated in the sense of special relativity.)

3. Arrow of time: The properties of an ensemble are defined completely by the preparation conditions. In particular, the distribution of "possessed values" of a given variable obtained for the subensemble on which we actually measure that variable is identical to their distribution in the complete ensemble.

The proof of Bell's theorem is now almost trivial. Consider the experimental setup described at the beginning of this lecture, with the relative orientations a, b, c, d so far unspecified:

- Step 1. By postulate 1, for any given pair of photons, the quantities A, B, C, D exist and each take either the value $+1$ or the value -1, irrespective of whether any of them are measured.

- Step 2. By postulate 2, the value of (e.g.) A for a given photon 1 cannot be affected by whether it is C or D, which is measured on photon 2 (in the most spectacular experiments, the "events" of choosing whether to measure C or D, and passing or not of polarizer a, are spacelike-separated in the sense of SR). Thus, for any given pair, the products AC, AD exist *with A having the same value in each.* Similarly for BC, BD (and AC, BC, etc.).

- Step 3. By simple exhaustion of the 16 (2^4) possibilities, the quantity $K \equiv AC + AD + BC - BD$ exists for any given pair and satisfies the inequality $-2 \leq K \leq 2$.

- Step 4. Hence the average value $\langle K \rangle_{\text{ens}}$ of K for any given ensemble (and in particular for the complete ensemble of photon pairs) exists and satisfies the inequality $-2 \leq \langle K \rangle_{\text{ens}} \leq 2$.

- Step 5. By postulate (3), each of the expectation values ($\langle AC \rangle$, etc.) has the same value for the subensensemble on which A is measured on photon 1 and C on photon 2, as for the photon pair ensemble as a whole. Thus, the quantity K_{exp} defined by

$$K_{\text{exp}} \equiv \langle AC \rangle_{\text{exp}} + \langle AD \rangle_{\text{exp}} + \langle BC \rangle_{\text{exp}} - \langle BD \rangle_{\text{exp}}$$

where, e.g.,

$$\langle AC \rangle_{\text{exp}} \overset{N_{ac}\to\infty}{\equiv} \frac{1}{N_{ac}} \sum_{i}^{(ac)} A_i C_i$$

also obeys $-2 \leq K_{\text{exp}} \leq 2$.

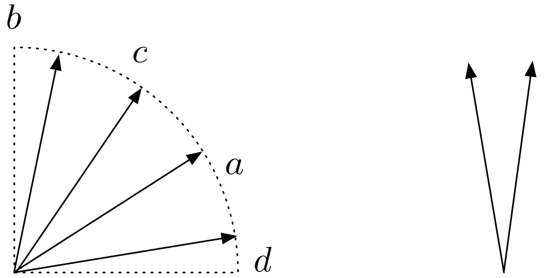
Thus we reach the conclusion:

> In any objective local theory, the experimentally measured quantity K_{exp} must satisfy the inequality
>
> $$-2 \leq \langle K \rangle_{\text{exp}} \leq 2$$

The above inequality, which is a slight generalization of Bell's original one, is usually known as the "CHSH inequality".

Before proceeding, two points. First, let's stress how "natural" and "commonsensical" the three defining postulates of an objective local theory look. We also note that we can relax the implicitly deterministic element in postulate 1, replacing it with the concept of a "state" of photon 1 in which a, etc., is only statistically determined, provided only that the probability distribution for a, and hence its expectation value for the state, does not depend on whether it is C or D that is measured on 2. The proof essentially goes through with A, etc., replaced by $\overline{A}$. (Bell game.)

Now the crunch: The QM predictions do not satisfy Bell's inequality! E.g., take



$$= \quad 22\frac{1}{2}^\circ$$

Then Bell's inequality states:

$$3p\left(22\frac{1}{2}^\circ\right) - p\left(67\frac{1}{2}^\circ\right) \leq 2$$

But the QM prediction is

$$p\left(22\frac{1}{2}^\circ\right) = \cos 45^\circ = \frac{1}{\sqrt{2}} : p\left(67\frac{1}{2}^\circ\right) = \cos 135^\circ = -\frac{1}{\sqrt{2}}$$

and thus the left-hand side of the equation is $2\sqrt{2}$, which is clearly greater than 2! Thus, in the situation considered, *no objective local theory can give experimental predictions identical to those of QM.*

So the obvious question is: what does experiment in fact say? The first experiment of this type was carried out by Freedman and Clauser in 1972, and since then there have been dozens of similar experiments, including the series by Aspect and coworkers (1983) in which the "events" of determining which of the two possible polarizations to measure on each photon, and also and also those of passing or not-passing the respective polarizers, were spacelike-separated. In some recent experiments, the spatial separation between the "detection" stations has been as large as 100 km. All of these experiments save one[††] have produced results that, given the quoted error bars, are consistent with the predictions of QM and violate the CHSH inequality – in some cases by as much as 80 standard deviations. Thus, it seems that nature does not believe in objective local theories! Two brief comments on the correlated-photon experiments:

1. Although in the Aspect experiment the "events" of the two photons of a pair passing or not passing the relevant polarizers were, as noted, (mostly) spacelike-separated in the sense of special relativity, the macroscopic events of production of a current pulse from the cathode of the associated photomultipliers (detectors) were probably not (or at least, not sufficiently for us to be able to apply Bell's inequality directly to these events). Why this may be relevant will become clear in the next lecture.

2. Although it is not particularly obvious at first sight, one way of looking at Bell's inequality is as a special and spectacular case of Gleason's theorem (lecture 19): the "space" of the polarization of each photon has two dimensions, so the "space" of the pair has $2 \times 2 = 4$ dimensions, and Gleason's theorem should apply to it.

[††]The one exception was repeated by another experimenter and found to give results compatible with the QM predictions: it is generally believed that the reasons why the initial version failed to do so are understood.

Appendix: Transformation of 2-photon states.

Suppose a single-photon state is polarized in direction x (so denoted $|\uparrow\rangle$.) We now express it in terms of states polarized parallel and perpendicular to direction x', which we denote $|\uparrow\rangle'$ and $|\rightarrow\rangle'$ respectively:

$$|\uparrow\rangle = \cos\theta|\uparrow\rangle' + \sin\theta|\rightarrow\rangle' \tag{1}$$

where θ is the angle by which x' is rotated relative to x. (We know (1) is right because this is just the way in which the electric field components transform between these two sets of axes: $E_x = E_{x'}\cos\theta + E_{y'}\sin\theta$). In a similar way, a photon state $|\rightarrow\rangle$ polarized perpendicular to x is expressed in terms of $|\uparrow\rangle'$ and $|\rightarrow\rangle'$ as

$$|\rightarrow\rangle = -\sin\theta|\uparrow\rangle' + \cos\theta|\rightarrow\rangle' \tag{2}$$

(note the $-$ sign, which is crucial in what follows). Now, the 2-photon state $|\uparrow\uparrow\rangle$ simply means that both photons are polarized parallel to x, so it can equally well be written as the "product" of $|\uparrow\rangle$ for 1 and $|\uparrow\rangle$ for 2. If now we re-express this state in terms of the states $|\uparrow\rangle'$ and $|\rightarrow\rangle'$, we have

$$\begin{aligned}
|\uparrow\uparrow\rangle &\equiv |\uparrow\rangle_1|\uparrow\rangle_2 = (\cos\theta|\uparrow\rangle_1' + \sin\theta|\rightarrow\rangle_1')(\cos\theta|\uparrow\rangle_2' + \sin\theta|\rightarrow\rangle_2') \\
&= \cos^2\theta|\uparrow\rangle_1'|\uparrow\rangle_2' + \sin^2\theta|\rightarrow\rangle_1'|\rightarrow\rangle_2' + \sin\theta\cos\theta(|\uparrow\rangle_1'|\rightarrow\rangle_2' + |\rightarrow\rangle_1'|\uparrow\rangle_2') \\
&\equiv \cos^2\theta|\uparrow\uparrow\rangle' + \sin^2\theta|\rightarrow\rightarrow\rangle' + \sin\theta\cos\theta(|\uparrow\rightarrow\rangle' + |\rightarrow\uparrow\rangle')
\end{aligned} \tag{3}$$

By similar algebra, the state $|\rightarrow\rightarrow\rangle$ becomes

$$|\rightarrow\rightarrow\rangle = \sin^2\theta|\uparrow\uparrow\rangle' + \cos^2\theta|\rightarrow\rightarrow\rangle' - \sin\theta\cos\theta(|\uparrow\rightarrow\rangle' + |\rightarrow\uparrow\rangle') \tag{4}$$

Now consider a quantum superposition of the states $|\uparrow\uparrow\rangle$ and $\rightarrow\rightarrow\rangle$ with coefficients $c_{\uparrow\uparrow}$ and $c_{\rightarrow\rightarrow}$ respectively:

$$\psi_{2\gamma} = c_{\uparrow\uparrow}|\uparrow\uparrow\rangle + c_{\rightarrow\rightarrow}|\rightarrow\rightarrow\rangle \tag{5}$$

By equations (3) and (4), when expressed in the $x'y'$ axes this is

$$\begin{aligned}
&(c_{\uparrow\uparrow}\cos^2\theta + c_{\rightarrow\rightarrow}\sin^2\theta)|\uparrow\uparrow\rangle' + (c_{\uparrow\uparrow}\sin^2\theta + c_{\rightarrow\rightarrow}\cos^2\theta)|\rightarrow\rightarrow\rangle' \\
&\quad + (c_{\uparrow\uparrow} - c_{\rightarrow\rightarrow})\sin\theta\cos\theta(|\uparrow\rightarrow\rangle' + |\rightarrow\uparrow\rangle')
\end{aligned} \tag{6}$$

If we now postulate that when measured in the new $x'y'$ axes, the two photons are always polarized either both parallel to x' or both perpendicular to x' (i.e., parallel to y'), this implies[‡‡] that the coefficients of the states $|\uparrow\rightarrow\rangle'$ and $\rightarrow\uparrow\rangle'$ must vanish, i.e.:

$$c_{\uparrow\uparrow} = c_{\rightarrow\rightarrow} \tag{7}$$

[‡‡]Except when either $\sin\theta = 0$ or $\cos\theta = 0$: the first choice simply corresponds to keeping the old axes and the second to interchanging them, so these are not of interest.

as stated earlier. Inserting this relation into (5), we see that in the original xy axes the 2-photon state in question is

$$\psi_{2\gamma} = \frac{1}{\sqrt{2}} \left(|\uparrow\uparrow\rangle + |\rightarrow\rightarrow\rangle \right) \tag{8}$$

(where the $1/\sqrt{2}$ follows from the fact that the probabilities of xx and yy polarization must add to 1). Similarly, substituting (7) into (5) and using the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$, we find that for *any* relative angle θ between x and x' the form of $\psi_{2\gamma}$ is

$$\psi_{2\gamma} = \frac{1}{\sqrt{2}} \left(|\uparrow\uparrow\rangle' + |\rightarrow\rightarrow\rangle' \right) \tag{9}$$

Thus, the 2-photon wave function has the same form when viewed from *any* choice of axes!