Space, Time and Simultaneity

Recall that

(a) in Newtonian mechanics ("Galilean space-time"): *time* is universal and is agreed upon by all observers; *spatial position* is relative to a reference frame: while for Newton there was a single "canonical" frame, for Galileo there is a whole class of such (the inertial frames).

A *Galilean transformation* between two inertial frames* preserves the laws of mechanics:

$$x' = x - vt,\ t' = t \tag{1}$$

so that the acceleration is the same in the two frames† and "acceleration = force/mass" remains an invariant statement (provided, of course, that force and mass are invariant under the transformation).

(b) in Maxwells' electromagnetic theory: electromagnetic waves (light) travel in free space at a speed, c($\cong 3 \times 10^8$ m/sec) which is independent of the wavelength (cf. SN 1987!) and of the motion of the source (This is theory, but can be verified experimentally: cf. the motion of Jupiter's moons.).

Under Galilean transformation, the velocity of light is not constant:

$$c \to c - v \tag{2}$$

(We can obtain this result either by following a "pulse" of light which should behave exactly like a mechanical particle or from the fact that $c = \lambda\nu\ :\ \nu' = \nu(1 - v/c)$, $\lambda' = \lambda$.) Thus, we can detect the "correct" frame by testing the isotropy of velocity of light.

What is light a wave of? The nineteenth-century answer was the aether. Thus the "canonical" frame is that in which the aether is at rest (and we are back to Newton!)

Now the Earth's frame is not an inertial one (both because of daily rotation and of the motion around the sun), and the most obvious hypothesis is that the aether is at rest in an inertial frame, most plausibly that of the fixed stars. If that is so, then we should be able to detect the motion of the Earth relative to the aether.

The Michelson-Morley Experiment

(1887 Michelson, Case, Nobel Prize)

This experiment was designed to measure the velocity of the Earth with respect to the aether. It was an interference experiment using visible light:

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*Unless otherwise stated, it is assumed throughout this lecture that relative velocities are always taken along the x-axis.

†[In the technical language of differential calculus, $d^2x'/dt'^2 = d^2x/dt^2$.]

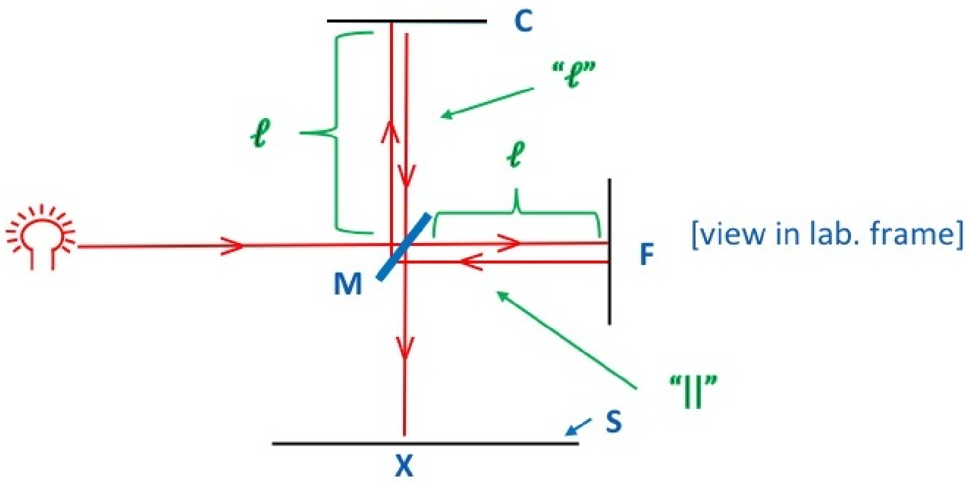


Figure 1: Schematic of Michelson-Morley

The experimenters looked at the pattern of the interference fringes on the screen S. Consider for definiteness the "symmetric" point X, and suppose for the moment the two "arms" are exactly equal in length. Then, if the apparatus is at rest with respect to the aether, the two beams interfere constructively at X, i.e., the crests arrive simultaneously (each takes $2l/c$ to get back to the mirror). Now suppose the apparatus is moving with respect to the aether with velocity v. Evidently the time taken in the "forward" arm ($M \to F \to M$) is (see fig. 2a):

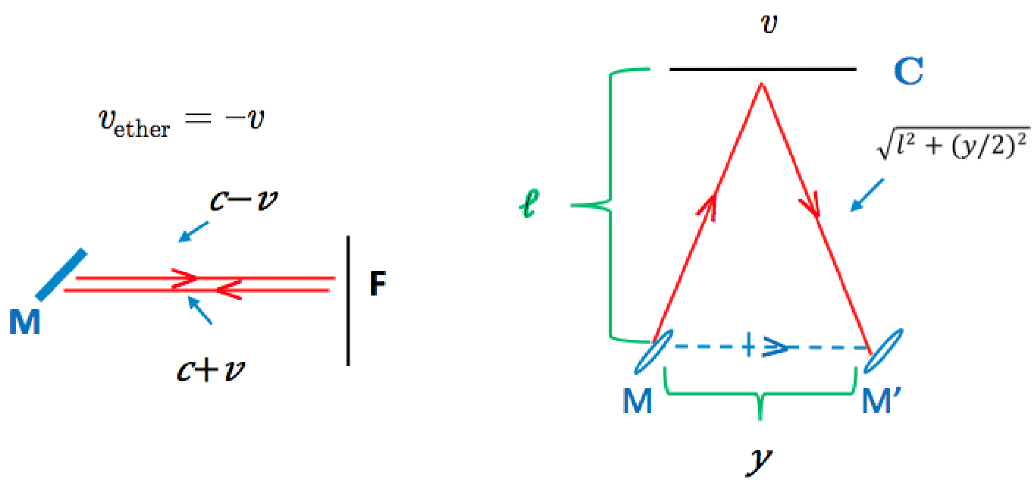
$$T_{\|} = l/(c-v) + l/(c+v) = 2lc/(c^2 - v^2) = \frac{2l}{c}\frac{1}{1 - v^2/c^2} \tag{3a}$$

What about the time $T_\perp$ taken in the "sideways" arm ($M \to C \to M$)? At first sight this should be simply $2l/c$. But this is not quite right, because in the time taken for the light to travel from M to M and back the mirror has moved a distance y relative to the aether, so that as viewed from the latter (see fig. 2b), the path is by a straightforward bit of geometry $2\sqrt{l^2 l + (y^2/4)}$(see figure); so $T_\perp = 2\sqrt{l^2 + \frac{y^2}{4}}/c$, but since $y = vT_\perp$, $T_\perp^2 = 4(l^2/c^2 + (v^2/4c^2)T_\perp^2)$ and so

$$T_\perp = \frac{2l}{c} \cdot \frac{1}{\sqrt{1 - v^2/c^2}} \tag{3b}$$

which is different from eq. (3a). Thus the time-delay between return to the mirror by the paths (and hence between arrival of the two beams at X) is

$$\Delta T = \frac{2l}{c}\left[\frac{1}{1 - v^2/c^2} - \frac{1}{\sqrt{1 - v^2/c^2}}\right] \tag{4}$$



$$T_{\parallel} = l/(c-v) + l/(c+v)$$

$$= 2l/c \cdot 1/(1 - v^2/c^2)$$

$$\begin{cases} y = v T_{\perp} \\ T_{\perp} = 2\sqrt{l^2 + (y/2)^2}/c \end{cases}$$

$$\Rightarrow T_{\perp}{}^2 = 4(l^2/c^2 + (v^2/4c^2)\,T_{\perp}{}^2\,)$$

$$\Rightarrow T_{\perp} = \frac{2l}{c} \cdot \frac{1}{\sqrt{1 - v^2/c^2}}$$

Figure 2: (a) (view from lab frame) (b) (view from frame of aether)

(For $v \ll c$, the realistic case, expression (4) turns out to be well approximated by lv^2/c^3, an approximation used to get eqn. (5) below.) Suppose we could choose a value of v, (call it v_0) such that ΔT is equal to exactly half a period; then the waves traveling by the two paths would exactly cancel, and we would get a dark region at X. So the illumination observed at X would change from light to dark as we vary the velocity of the apparatus relative to the aether from 0 to v_0, and in this way we could detect motion relative to the aether.

Although this is the basic principle of the Michelson-Morley experiment, the real life implementation is a bit more complicated. In the first place, the assumption that the two arms of the interferometer are of exactly equal length is experimentally unrealistic; second, the maximum value of v attainable is only about half of the v_0 just defined, so that we could not in practice change the illumination at X much from its original value, and this is not a practical way of seeing the effect; and third, because of the Earth's motion around the Sun, the laboratory is anyway never at rest with respect to the aether. A more careful consideration leads to the following result: were the Earth stationary with respect to the aether, there should appear a pattern of bright and dark lines on the screen S with spacing d due to the interference of light traveling by the two paths. When the apparatus moves relative to the aether in the direction shown as horizontal in the diagram, this "fringe pattern" should shift by an amount fd, where the fraction f is given approximately by

$$f = (l/\lambda)(v^2/c^2) \tag{5}$$

where λ is the wavelength of the light. If it moves transversely (i.e., in the vertical direction in the figure) relative to the aether, the shift should be reversed.

Now the Earth moves around the Sun at a speed of approximately 30 km/sec (and rotates, in the latitude of Cleveland, at approximately 0.3 km/sec, so this rotation is negligible by comparison) and the diurnal rotation carries the apparatus around so that it is sometimes parallel and sometimes perpendicular to this velocity, so there should be a total oscillation of the fringes with a 12-hour period and an amplitude given by eqn. (5). The quantity $(l/\lambda)(v^2/c^2)$ is $\sim (6\text{m}/5 \times 10^{-7}\text{m})(30/3 \times 10^5)^2 \sim 0.1$, so with careful observation the "fringe shift" should have been visible. But none was seen (and there were many confirmations of this result over the next few decades).

Possible explanations:

1. The Earth is an inertial frame and therefore the aether is at rest with respect to it (i.e., back to Ptolemy!) In view of the overwhelming evidence of the previous three centuries to the contrary, this hypothesis has not to my knowledge been seriously considered.

2. The Earth is not an inertial frame, but drags the aether with it locally. Two arguments tell against this hypothesis: first, it had been shown by Fizeau that the "apparent" velocity of light in a moving medium such as water is $\frac{c}{n} - v(1 - \frac{1}{n^2})$, where n is the so-called "refractive index" and v is the velocity of the medium in the direction of the light; since for the aether (vacuum!) n is by definition 1, this means that the velocity of light would remain isotropic as viewed from the inertial frame. Second, the latter conclusion is supported by the phenomenon of stellar aberration (cf. Rohrlich p. 53): the light from a star seems to come from a slightly different direction depending on whether the Earth is moving towards or away from it.

3. Lorentz contraction hypothesis: Recall the "toy" problem where with the apparatus at rest $l_\parallel = l_\perp \equiv l$. Suppose that when it moves relative to the aether $l_\parallel$ and (possibly) $l_\perp$ changes. Then eqns. (3a) and (3b) are respectively replaced by

$$T_\perp = \frac{2l_\perp}{c} \frac{1}{\sqrt{1 - v^2/c^2)}}, \qquad T_\parallel = \frac{2l_\parallel}{c} \frac{1}{1 - v^2/c^2} \tag{6}$$

when $l_\parallel$ and $l_\perp$ are the lengths of the "forward" and "sideways" arms respectively. How much would $l_\parallel$ have to change by to make $T_\perp$ always equal to $T_\parallel$, assuming that $l_\perp$ remains equal to l?

Answer:

$$l_{\|}(v) = l_{\perp} \cdot \sqrt{1 - v^2/c^2} \qquad ! \tag{7}$$

The Irish physicist George Fitzgerald and the Dutch physicist Henrik Lorentz postulated that because of the way that the electrical forces holding bodies together supposedly change when the body moves relative to the ether, such a physical contraction (the "Lorentz-Fitzgerald contraction") indeed occurs.

The Michelson-Morley experiment shows that *one particular* experiment designed to detect motion of the Earth with respect to the aether won't work. Suppose we jump from this to the more general hypothesis (later very adequately confirmed in experiment) that *no* experiment will ever be able to detect motion with respect to the aether. Keeping for the moment our conventional ideas about space and time, what else (apart from the Lorentz contraction) would this imply?

Recall that in the Michelson-Morley setup the time delay in the perpendicular arm was $T_{\perp} = \frac{2l}{c}\frac{1}{\sqrt{1-v^2/c^2}}$. We have postulated (at least for the moment) that there is no Lorentz contraction in this arm. But then, if we compare the time taken when we are moving with respect to the aether with that taken when stationary, we should see a difference! (Note: We assume here we have an absolute measure of $T_{\perp}$, not just of the difference $T_{\perp} - T_{\|}$. This is experimentally difficult, but there are no objections of principle). If in fact we see no difference, then the only way out must be that *our clocks are running slow* (with respect to clocks stationary relative to the aether) by a factor $\sqrt{1 - v^2/c^2}$. (This phenomenon is called the *Fitzgerald time dilation.*)

(Note: We may reasonably ask whether a more general length contraction hypothesis would be adequate with no time dilation. The answer is no. The argument is essentially the same as that given below in the context of special relativity.)

Thus, we can explain the results of the Michelson-Morley experiment, and indeed, as it turns out, of *all* experiments to date, if we use the following hypotheses:

1. (Lorentz contraction): Bodies moving with velocity v with respect to the aether are physically contracted along the direction of motion by a factor $\sqrt{1 - v^2/c^2}$.

2. (Fitzgerald time dilation): Clocks moving with velocity v relative to the aether slow down by a factor $\sqrt{1 - v^2/c^2}$.

But is this necessary?

Enter Einstein

He asked: Suppose we *postulate* that the speed of light as measured by all inertial observers is the same. What does this imply? Clearly it is incompatible with the Galilean transformation for the space-time coordinates of events.

$$x' = x - vt \qquad t' = t \tag{8}$$

Now it is difficult to change the first very much, because v is by definition the velocity of the moving frame with respect to the original one. At best we can write, perhaps,

$$x' = \text{const.}(x - vt) \tag{9}$$

where the constant might be a function of v. So we must challenge the assumption $t' = t$, i.e., that the time of events as detected by different observers must be the same and in particular the idea that *observers using different rest frames can agree about simultaneity.* Two observers using reference frames in relative motion with respect to one another do not in general agree about whether two events which happen at different times occur at the same place (see e.g., the ping-pong example given by Hawking): why should the converse be true? How do we establish the "simultaneity" of two events? Of course, if they take place at the same place in space in any reference frame *and* are simultaneous then they must be simultaneous for all observers. But, in general, we need to set up a system of synchronized clocks. How to do this?

Proposal: synchronize clocks at the same point and transport them. But, according to Fitzgerald running clocks go slow! Now, *if* I am at rest with respect to the aether, this doesn't matter: the rate of slowing is proportional to v^2/c^2, and if I want to put the clock eventually a distance Δx away, I can take a transit time Δt: then $v = \Delta x/\Delta t$, the rate of slowing $= v^2/c^2$ and lasts a time $\Delta t \sim \Delta x/v$, so the total error δt is $\sim (v^2/c^2)(\Delta x/v) \sim (v/c^2)\Delta x$ and I can make this as small as I please by taking v small enough. But suppose I am *already* in motion with respect to the aether with velocity u. Then my clock is *already* going slow by a factor $\sqrt{1 - u^2/c^2} \approx 1 - \frac{1}{2}(\frac{u^2}{c^2})$. When I start moving it with velocity v, it goes slow by $1 - \frac{1}{2}(u + v)^2c^2$, so the *extra* slowing is approximately uv/c^2 for small v. Again we have $\Delta t = \Delta x/v$ so the total slowing is $\delta t \approx u\Delta x/c^2$. This *cannot* be made negligibly small by taking $v \to 0$!

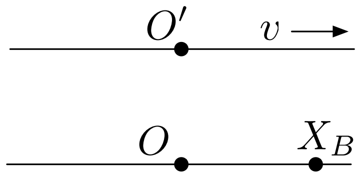
Of course, if I am a believer in the aether and believe I know my velocity with respect to it, I can correct for this effect. (This "correction" would be equivalent to choosing a definition of "absolute" simultaneity as meaning simultaneity as judged by an observer in the frame of the aether!) But suppose I don't? Einstein's proposal: Suppose we *make it a fundamental assumption* that the speed of light as seen by all inertial observers is isotropic and has a unique value c. Then we can *define* the time of a distant event as follows: imagine sending out a light signal so that it reaches the distant point exactly when the event occurs†, and is reflected. If t_{em} and t_{ret} are the times of emission and return (reabsorption) as measured by my "local" clock, then *by definition* the time of the event is $(t_{\text{em}} + t_{\text{ret}})/2$.

But with this definition events which are simultaneous for one inertial observer are clearly not so for other inertial observers!

(The easiest way to see this is by reductio ad absurdum (see figure): consider the three events of emission of a light signal by O', reflection at space point X_B and reabsorption by O'. O' will by hypothesis judge these events to be equally spaced in time: but O cannot, since he reckons that since O' has moved, the time taken by the light on the

† I could verify this, e.g., by a later report from my friend stationed there

return trip is less than that on the outward one. Thus, they cannot assign the same time coordinates to all three events.)



What are the *quantitative* relations between coordinates and times as judged by different observers?

Consider two events which, as judged by observer O, are separated by Δx in space and Δt in time. Consider an observer O' moving in the positive Δx direction with respect to O with velocity v. For the reason given earlier we expect that the most general acceptable relation between $\Delta x'$ and Δx, Δt is

$$\Delta x' = f(v)(\Delta x - v\Delta t) \tag{10}$$

note that this is linear in Δx, Δt. Since $\Delta t' = \Delta t$ for $v = 0$, it seems reasonable to try

$$\Delta t' = g(v)(\Delta t + A \cdot v\Delta x) \tag{11}$$

with $g(0) = 1$.

We now determine the quantities $f(v)$, $g(v)$, A from the considerations (a) that the constancy of the speed of light requires

$$\frac{\Delta x'}{\Delta t'} = \frac{\Delta x}{\Delta t} \qquad \text{when} \qquad \frac{\Delta x}{\Delta t} = c \tag{12}$$

(i.e., the two events are connected by a light wave.) This yields

$$f(v)(1 - v/c) = g(v)(1 + Avc) \tag{13}$$

(b) that if $\Delta x' = f(v)(\Delta x - v\Delta t)$ then by reciprocity

$$\Delta x = f(-v)(\Delta x' + v\Delta t') \tag{14}$$

which gives (substituting for $\Delta x'$ and $\Delta t'$)

$$\Delta x = f(-v)\left[f(v)\Delta x - vf(v)\Delta t + vg(v)\Delta t + v^2 Ag(v)\Delta x\right] \tag{15}$$

i.e., the two conditions*

$$f(v) = g(v) \tag{16}$$
$$f(-v)[f(v) + v^2 Ag(v)] = 1 \tag{17}$$

*Since Δx and Δt can be chosen independently, the coefficient of Δt in eqn. (15) must vanish.

Combining equations (13) and (16-17) and assuming $f(v) = f(-v)$ (since no "unique" sense of velocity is picked out!) gives the unique solution $A = \frac{-1}{c^2}$, $f(v) = g(v) = (1 - v^2/c^2)^{-1/2} \equiv \gamma(v)$. Thus:

$$\boxed{\Delta x' = \frac{(\Delta x - v\Delta t)}{\sqrt{1 - v^2/c^2}}} \tag{18}$$

$$\boxed{\Delta t' = \frac{(\Delta t - v\Delta x/c^2)}{\sqrt{1 - v^2/c^2}}} \tag{19}$$

Similar arguments give $\Delta y' = \Delta y$, $\Delta z' = \Delta z$, i.e., spatial intervals transverse to the relative velocity of the two frames are not affected. This is the famous *Lorentz transformation*. Note one immediate consequence: If we define

$$\Delta s^2 \equiv c^2 \Delta t^2 - \Delta x^2 \tag{20}$$

then $\Delta s'^2 = \Delta s^2$. Thus although neither the space interval near the time interval between two events is independent of the observer, the "space-time interval" Δs is! Minkowski: "Henceforth space by itself or time by itself are doomed to fade away into shadows, and only a kind of union of the two will possess an independent reality."

[Reception of Einstein's work.]