Examples of Complex Sound Fields (Continued):

**Example #3: Point Monopole Sound Source – Spherical Waves Propagating in “Free Air”:**

Note: no electromagnetic analog exists for this acoustic example, due to the manifest vectorial nature of the $EM$ field – which is mediated at the microscopic level by the spin-1 photon. So-called electric monopole $\{E(0)\}$ and/or magnetic monopole $\{M(0)\}$ radiation associated e.g. with a spherically-symmetric, radially oscillating electric charge distribution $\rho_e(\vec{r},t) = q_o \delta^3(\vec{r}) e^{i\omega t}$ and/or magnetic charge distribution $\rho_m(\vec{r},t) = g_o \delta^3(\vec{r}) e^{i\omega t}$ cannot occur.

Imagine a spherically-symmetric, point sound source located at the origin of coordinates $\vec{r} = 0$ that isotropically emits monochromatic spherical acoustic waves into “free air”. The 3-D wave equation describing the behavior of the instantaneous/physical $\{i.e. purely real time-domain\}$ over-pressure $p(\vec{r},t)$ at the space-time point $(\vec{r},t)$ is an inhomogeneous, linear 2nd-order differential equation:

$$\nabla^2 p(\vec{r},t) - \frac{1}{c^2} \frac{\partial^2 p(\vec{r},t)}{\partial t^2} = -4\pi B_o \delta^3(\vec{r}) \cos \omega t$$

The gradient $\nabla$ and Laplacian $\nabla^2 \equiv \nabla \cdot \nabla$ operators in 3-D spherical-polar $(r, \theta, \phi)$ coordinates are:

$$\nabla = \frac{\partial}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \hat{\phi}$$

and:

$$\nabla^2 \equiv \nabla \cdot \nabla = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

The RHS of this 3-D wave equation is originates from $\nabla^2 (1/r) = -4\pi \delta^3(\vec{r})$, thus $-4\pi B_o \delta^3(\vec{r}) \cos \omega t$ describes the point sound source located at the origin $\vec{r} = 0$, radiating sound isotropically into $4\pi$ steradians. The function $\delta^3(\vec{r})$ is known as {Dirac’s} 3-D delta function, which has many intriguing mathematical properties, one of which is that the 3-D delta function $\delta^3(\vec{r})$ has an (infinite!) spike at the origin $\vec{r} = 0$ and is $\equiv 0$ elsewhere. Thus, we can equivalently write $\delta^3(\vec{r}) = \delta^3(\vec{r} - 0)$. Note that in spherical-polar coordinates $(r, \theta, \phi)$, that $\delta^3(\vec{r}) = \frac{1}{4\pi r^2} \delta(r)$ where $\delta(r) = \delta(r - 0)$ is the 1-D delta function in the radial ($r$) direction only. If we integrate the 3-D delta function over a volume $V$ containing the origin $\vec{r} = 0$, e.g. integrate over all space:

$$\int_V \delta^3(\vec{r}) \, dV = \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \int_{r=0}^{r=\infty} \delta^3(\vec{r}) \cdot r^2 \sin \theta \, d\theta \, d\phi = \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \int_{r=0}^{r=\infty} \frac{1}{4\pi r^2} \delta(r) \cdot r^2 \sin \theta \, d\theta \, d\phi$$

$$= \frac{1}{4\pi} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \int_{r=0}^{r=\infty} \delta(r) \cdot r^2 \sin \theta \, d\theta \, d\phi = \frac{1}{2\pi} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=\infty} \delta(r) \cdot r \, d\theta \, d\phi$$

$$= \frac{1}{2} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=\infty} \delta(r) \cdot r \cdot d\cos \theta = \frac{1}{2} \int_{\theta=-\pi}^{\theta=\pi} \int_{r=0}^{r=\infty} \delta(r) \cdot r \cdot d\theta = \frac{1}{2} \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=\pi} \delta(r) \cdot r \, d\theta \, dr = 1$$

If the volume $V$ does not contain the origin $\vec{r} = 0$, then:

$$\int_V \delta^3(\vec{r}) \, dV = 0.$$
Note also that the 3-D \{1-D\} delta functions $\delta^3(\vec{r}) \{ \delta(r) \}$ have SI units of $m^{-3} \{ m \}$, respectively; the integrals $\int_V \delta^3(\vec{r})dV = 1$ and $\int_{r=0}^{r=\infty} \delta(r)dr = 1$ respectively, are dimensionless.

If we now integrate the above inhomogeneous 2nd order linear differential equation over a \{finite\} arbitrary volume $V$ but e.g. centered on, and thus containing the origin $\vec{r} = 0$, where the isotropic point sound source is located:

$$\int_V \nabla^2 p(\vec{r},t)dV - \frac{1}{c^2} \int_V \frac{\partial^2 p(\vec{r},t)}{\partial t^2}dV = -4\pi B_o \int_V \delta^3(\vec{r})dV \cos \omega t$$

Then using the Gauss divergence theorem: $\int_V \nabla^2 p(\vec{r},t)dV = \int_S \nabla \cdot \nabla p(\vec{r},t)dV = \int_S \nabla p(\vec{r},t) \cdot \hat{n}dS$

where $\hat{n}$ is the outward-pointing unit normal to the surface $S$ \{which encloses/bounds the volume $V$\} and: $\int_V \delta^3(\vec{r})dV = 1$, the above integral relation then becomes:

$$\int_S \nabla p(\vec{r},t) \cdot \hat{n}dS - \frac{1}{c^2} \int_V \frac{\partial^2 p(\vec{r},t)}{\partial t^2}dV = -4\pi B_o \cos \omega t$$

A spherically-symmetric time-dependent scalar $p(\vec{r},t)$ over-pressure has rotational invariance/rotational symmetry and therefore cannot have any explicit $\theta$- and/or $\phi$-dependence – only $r$-dependence. Thus $p(\vec{r},t) = p(r,t) \neq f(\theta, \phi)$, and hence for a spherically-symmetric point sound source located at the origin of coordinates, then:

$$\nabla p(\vec{r},t) = \nabla p(r,t) = \left\{ \frac{\partial}{\partial r} \vec{r} + \frac{1}{r} \frac{\partial}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \hat{\phi} \right\} p(r,t) = \frac{\partial p(r,t)}{\partial r} \hat{r}$$

and the 3-D integral wave equation for the scalar over-pressure $p(\vec{r},t)$ associated with this isotropic point sound source (for $r > 0$) becomes:

$$\int_S \frac{\partial p(\vec{r},t)}{\partial r} \hat{r} \cdot \hat{n}dS - \frac{1}{c^2} \int_V \frac{\partial^2 p(\vec{r},t)}{\partial t^2}dV = -4\pi B_o \cos \omega t$$

The instantaneous/physical \{i.e. purely real time-domain\} solution to this integral wave equation is a purely real spherical-outgoing harmonic over-pressure wave:

$$p(r,t) = \frac{B_o}{r} \cos(\omega t - kr) \ (Pascals) \ . \ \text{Note that the constant } B_o \ \text{has SI units of Pascal-m} \ .$$

The instantaneous/physical \{i.e. purely real time-domain\} particle velocity $\vec{u}(\vec{r},t)$ associated with this problem is determined via use of the \{linearized\} Euler’s equation for inviscid fluid flow:

$$\frac{\partial \vec{u}(\vec{r},t)}{\partial t} = -\frac{1}{\rho_o} \nabla p(\vec{r},t)$$
Since $p(\hat{r}, t) = p(r, t) \neq fcn(\theta, \phi)$, then $\nabla p(\hat{r}, t) = \nabla p(r, t) = (\partial p(r, t)/\partial r) \hat{r}$ and thus the instantaneous/physical (i.e. purely real time-domain) particle velocity $\vec{u}(\hat{r}, t)$ associated with a spherically-symmetric monochromatic point sound source can only be oriented in the radial direction, i.e. $\vec{u}(\hat{r}, t) = u_r(r, t) \hat{r} \neq fcn(\theta, \phi)$:

$$\frac{\partial \hat{u}(\hat{r}, t)}{\partial t} = -\frac{1}{\rho_o} \nabla p(\hat{r}, t) \implies \frac{\partial u_r(r, t)}{\partial t} \hat{r} = -\frac{1}{\rho_o} \frac{\partial p(r, t)}{\partial r} \hat{r} \implies \frac{\partial u_r(r, t)}{\partial t} = -\frac{1}{\rho_o} \frac{\partial p(r, t)}{\partial r}$$

Now:

$$\frac{\partial p(r, t)}{\partial r} = \frac{\partial}{\partial r} \left\{ \frac{B_o}{r} \cos(\omega t - kr) \right\} = -\frac{B_o}{r^2} \cos(\omega t - kr) + \frac{k B_o}{r} \sin(\omega t - kr)$$

Then:

$$\frac{\partial u_r(r, t)}{\partial t} = -\frac{1}{\rho_o} \frac{\partial p(r, t)}{\partial r} = -\frac{1}{\rho_o} \left\{ -\frac{B_o}{r^2} \cos(\omega t - kr) + \frac{k B_o}{r} \sin(\omega t - kr) \right\}$$

$$= + \frac{B_o}{\rho_o r} \left\{ \frac{1}{r} \cos(\omega t - kr) - k \sin(\omega t - kr) \right\}$$

Thus:

$$u_r(r, t) = \frac{B_o}{\omega \rho_o r} \left\{ \frac{1}{r} \sin(\omega t - kr) + k \cos(\omega t - kr) \right\} = \frac{B_o k}{\omega \rho_o r} \left\{ \cos(\omega t - kr) + \frac{1}{kr} \sin(\omega t - kr) \right\} (m/s)$$

Using $c = \omega/k$ and $z_o = \rho_o c$ this relation becomes:

$$u_r(r, t) = \frac{B_o}{z_o r} \left\{ \cos(\omega t - kr) + \frac{1}{kr} \sin(\omega t - kr) \right\} (m/s)$$

We can then “complexify” the radial-outgoing spherical over-pressure and particle velocity waves:

$$p(r, t) = \frac{B_o}{r} \cos(\omega t - kr) \implies \bar{p}(r, t) = \frac{B_o}{r} \left\{ \cos(\omega t - kr) + i \sin(\omega t - kr) \right\} = \frac{B_o}{r} e^{i(\omega t - kr)}$$

and: $u_r(r, t) = \frac{1}{z_o r} \left\{ \cos(\omega t - kr) + \frac{1}{kr} \sin(\omega t - kr) \right\} \implies$

$$\bar{u}_r(r, t) = \frac{1}{z_o r} \left\{ \cos(\omega t - kr) + \frac{1}{kr} \sin(\omega t - kr) \right\} + i \left\{ \sin(\omega t - kr) - \frac{1}{kr} \cos(\omega t - kr) \right\}$$

$$= \frac{1}{z_o r} \left\{ \cos(\omega t - kr) + i \sin(\omega t - kr) \right\} - i \frac{1}{kr} \left\{ \cos(\omega t - kr) + i \sin(\omega t - kr) \right\}$$

$$= \frac{1}{z_o r} \left\{ e^{i(\omega t - kr)} \right\} - i \frac{1}{kr} \left\{ e^{i(\omega t - kr)} \right\} = \frac{1}{z_o r} \left\{ 1 - i \frac{1}{kr} e^{i(\omega t - kr)} \right\} = \frac{1}{z_o} \left\{ 1 - i \frac{1}{kr} \right\} \bar{p}(r, t)$$
The relationships between the complex time-domain and complex frequency-domain over-pressure and radial particle velocity amplitudes for the point monopole sound source are:

\[
\tilde{p}(r, t) = \frac{B_o}{r} e^{i(\omega - kr)} = \tilde{p}(r, \omega) e^{i\omega t} \quad \text{i.e.} \quad \tilde{p}(r, \omega) = \frac{B_o}{r} e^{-ikr}
\]

\[
\tilde{u}_r(r, t) = \frac{1}{z_o} \frac{B_o}{r} \left[ 1 - \frac{i}{kr} \right] e^{i(\omega - kr)} = \tilde{u}_r(r, \omega) e^{i\omega t} \quad \text{i.e.} \quad \tilde{u}_r(r, \omega) = \frac{1}{z_o} \frac{B_o}{r} \left[ 1 - \frac{i}{kr} \right] e^{-ikr}
\]

Note that the purely real “amplitude” \(B_o\) is in general complex \(\tilde{B}\), in order to accommodate an (arbitrary) absolute phase \(\varphi_B\), which we can always “rotate” away, simply by re-defining the zero of time: \(t \rightarrow t - (\varphi_B / \omega)\). So, to make life simpler, we can set \(\varphi_B = 0\), then \(\tilde{B} = |\tilde{B}| = B_o\) is purely real. The magnitudes of the complex time-domain over-pressure and radial particle velocity amplitudes then are:

\[
|\tilde{p}(r, t)| = \frac{|\tilde{B}|}{r} = |\tilde{p}(r, \omega)| \quad \text{and} \quad |\tilde{u}_r(r, t)| = \frac{1}{z_o} \frac{|\tilde{B}|}{r} \sqrt{1 + (1/kr)^2} = |\tilde{u}_r(r, \omega)|
\]

The phases of the complex pressure and radial particle velocity are:

\[
\varphi_p = \varphi_B = 0 \quad \text{and} \quad \varphi_{u_r} = \tan^{-1}(-1/kr) + \varphi_B = -\tan^{-1}(1/kr) + \varphi_B = -\tan^{-1}(1/kr)
\]

Thus, we also see that:

\[
\Delta \varphi_{p-u_r} \equiv \varphi_p - \varphi_{u_r} = \varphi_B - \left( -\tan^{-1}(1/kr) + \varphi_B \right) = +\tan^{-1}(1/kr) \quad ( = \varphi_z - \varphi_i)
\]

n.b. the phase difference \(\Delta \varphi_{p-u_r} = \varphi_p - \varphi_{u_r} \quad ( = \varphi_z - \varphi_i)\) is independent of the absolute phase \(\varphi_B\).

Let us examine the behavior of time-domain \(\tilde{p}(r, t)\) and \(\tilde{u}_r(r, t)\) as \(r\) increases from \(r = 0\). The complex over-pressure \(\tilde{p}(r, t) = (B_o/r) e^{i(\omega - kr)}\) simply decreases with increasing \(r\), modulated by the {complex} oscillatory factor \(e^{i(\omega - kr)}\). The behavior of \(\tilde{u}_r(r, t) = \frac{1}{z_o} \frac{B_o}{r} \left[ 1 - \frac{i}{kr} \right] e^{i(\omega - kr)}\) is mathematically more interesting... When \(r \approx 0\) \(\{i.e. \text{in proximity to the point sound source}\}\), then \(kr \ll 1\), and the imaginary part of \(\tilde{u}_r(r, t)\) dominates, because then \(|i/kr| = 1/kr \gg 1\), thus for \(r \approx 0\) \((kr \ll 1)\), \(\tilde{u}_r(r, t)\) will be \(\sim 90^\circ \text{ out-of-phase/in quadrature}\) with the complex over-pressure \(\tilde{p}(r, t)\), and hence in proximity to the point sound source, both the frequency domain complex radial specific acoustic impedance \(\tilde{z}_r(r, \omega) = \tilde{p}(r, \omega)/\tilde{u}_r(r, \omega)\) and the frequency domain complex radial acoustic intensity \(\tilde{I}_r(r, \omega) = \frac{1}{2} \tilde{p}(r, \omega) \cdot \tilde{u}_r(r, \omega)\) will be \textbf{largely imaginary/reactive/non-propagating} – i.e. for \(r \approx 0\) \((kr \ll 1)\), sound energy is largely stored locally, oscillating/sloshing back-and-forth during each cycle of oscillation...
However, far from the sound source, when \( r \to \infty \) \((kr \gg 1)\), then \(|k/r| = 1/kr \ll 1\) and \(\tilde{u}_r (r, t)\) is predominantly real, and also in-phase with the complex over-pressure \(\tilde{p} (r, t)\), and hence far from the point sound source, both the frequency-domain complex radial specific acoustic impedance \(\tilde{z}_{a_r} (r, \omega) = \tilde{p} (r, \omega)/\tilde{u}_r (r, \omega)\) and frequency-domain radial acoustic intensity \(\tilde{I}_{a_r} (r, \omega) = \frac{1}{2} \tilde{p} (r, \omega) \cdot \tilde{u}_r^* (r, \omega)\) will be largely real quantities associated with active/propagating sound radiation originating from the \{distant\} point sound source.

The “near-field” \((kr \ll 1)\) and the “far-field” \((kr \gg 1)\) behavior of the complex sound field \(\tilde{S} (r, t)\) associated with an isotropic point sound source emitting monochromatic spherical outgoing waves can be easily understood from a physically-intuitive perspective:

Far from the point sound source \((r \to \infty)\), the spherical waves are to an increasing degree nearly perfect plane waves – the curvature of the spherical wavefronts becomes increasingly neglectable as \(r\) increases far from the point sound source. As we have seen in the previous P406 Lecture Notes 12, for a monochromatic plane/traveling wave propagating in “free air”, the complex over-pressure \(\tilde{p} (r, t)\) and \{longitudinal\} particle velocity \(\tilde{u}^l (r, t)\) are perfectly in-phase with each other; hence \{here\} both the frequency domain complex radial specific acoustic impedance \(\tilde{z}_{a_r} (r, \omega) = \tilde{p} (r, \omega)/\tilde{u}_r (r, \omega)\) \((= \rho_c c \equiv z^l)\) and the frequency domain radial acoustic intensity \(\tilde{I}_{a_r} (r, \omega) = \frac{1}{2} \tilde{p} (r, \omega) \cdot \tilde{u}_r^* (r, \omega)\) will be purely real quantities associated with active/propagating sound radiation in the form of these monochromatic plane waves.

However, in proximity to the point sound source \((r \approx 0)\), the curvature of the spherical wavefronts becomes increasingly important as \(r \to 0\), one consequence of which is that the radial particle velocity \(\tilde{u}_r (r, t)\) becomes increasingly more and more imaginary/out-of-phase with the complex over-pressure \(\tilde{p} (r, t)\) as the curvature of the spherical wavefronts becomes more and more significant as \(r \to 0\).

In the “intermediate field” region associated with an isotropic point sound source, this is where \(kr \sim 1\), and the complex particle velocity \(\tilde{u}_r (r, t)\) has \(\sim\) equal real and imaginary components, and hence is \(\sim 45^\circ\) out of phase with the complex over-pressure \(\tilde{p} (r, t)\).

Thus, for a complex sound field \(\tilde{S} (r, t)\) associated with an arbitrary sound source, if the wavefronts are non-planar (such as would be expected in proximity to the sound source), the particle velocity \(\tilde{u} (r, t)\) will acquire an increasingly large imaginary component as the wavefronts become increasingly non-planar, \(\tilde{u} (r, t)\) will become increasingly disparate in phase with respect to the complex over-pressure \(\tilde{p} (r, t)\). Consequently, the complex specific acoustic impedance \(\tilde{z}_{a_r} (r, \omega)\) and complex acoustic intensity \(\tilde{I}_{a_r} (r, \omega)\) will acquire increasingly reactive/non-propagating/imaginary components as the wavefronts become increasingly non-planar, in proximity to the sound source…
The complex sound field \( \vec{S}(r,t) \) associated with a “point” monopole sound source radiating monochromatic, radially-outgoing spherical waves thus has \textbf{three} basic regions, or zones:

(a) **The “near” zone:** \( kr \ll 1 \), the radial particle velocity \( \vec{u}_r(r,t) \sim -i \left| \vec{B} \right|/z_o kr^2 \) is largely \textit{imaginary} (\textit{i.e.} \textit{reactive}), decreasing as \( \sim 1/r^2 \) (while the pressure \( \vec{p}(r,t) \sim \left| \vec{B} \right|/r \) decreases as \( \sim 1/r \) ), and where the particle velocity \textit{lags} the pressure by \( \varphi_p - \varphi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(\infty) \sim 90^\circ \) in phase. This region of the complex sound field \( \vec{S}(r,t) \) is \textit{reactive}, largely consisting of \textit{non-propagating} acoustic energy, and is also \textit{inertia}-like (\textit{i.e.} \textit{mass}-like), because \( u_o(r) = \text{Im}\{\vec{u}_r(r)\} \leq 0 \) for \( kr \ll 1 \).

(b) **The “intermediate” zone:** \( kr \sim 1 \), the radial particle velocity \( \vec{u}_r(r,t) \sim (1-i) \left| \vec{B} \right|/2z_o r \) has \~ comparable \textit{real} and \textit{imaginary} components, decreasing somewhat/slightly faster than \( \sim 1/r \) (while the pressure \( \vec{p}(r,t) \sim \left| \vec{B} \right|/r \) decreases as \( \sim 1/r \) ), and where the particle velocity \textit{lags} the pressure by \( \varphi_p - \varphi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(1) \sim 45^\circ \) in phase. This region of the complex sound field \( \vec{S}(r,t) \) is \~ an \textit{equal} mix of \textit{propagating} and \textit{non-propagating} acoustic energy.

(c) **The “far” (or radiation) zone:** \( kr \gg 1 \), the radial particle velocity \( \vec{u}_r(r,t) \sim 1 \left| \vec{B} \right|/z_o r \) is largely \textit{real} (\textit{i.e.} \textit{active}), decreasing \~ as \( \sim 1/r \) (while the pressure \( \vec{p}(r,t) \sim \left| \vec{B} \right|/r \) also decreases as \( \sim 1/r \) ), and where the radial particle velocity is \textit{in-phase} with the pressure, \textit{i.e.} \( \varphi_p - \varphi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(0) \sim 0^\circ \). This region of the complex sound field \( \vec{S}(r,t) \) is \textit{active}, dominated by \textit{propagating} acoustic energy.

The radial complex \textit{specific} acoustic impedance and its magnitude are:

\[
\vec{Z}_{a_r}(r,\omega) = \frac{\vec{p}(r,\omega)}{\vec{u}_r(r,\omega)} = z_o \frac{1}{1-i/kr} = z_o \left[ 1 + i/kr \right] \quad \text{and:} \quad \left| \vec{Z}_{a_r}(r,\omega) \right| = z_o \sqrt{\frac{1 + (1/kr)^2}{1 + (1/kr)^2}} = z_o \frac{1}{\sqrt{1 + (1/kr)^2}}
\]

Note that since \( \vec{Z}_{a_r}(r,\omega) = \rho_o \vec{c}_{a_r}(r,\omega) \) these relations can be written in dimensionless form as:

\[
\frac{\vec{Z}_{a_r}(r,\omega)}{z_o} = \left[ 1 + i/kr \right] \frac{1}{1 + (1/kr)^2} \quad \text{and:} \quad \frac{\vec{Z}_{a_r}(r,\omega)}{z_o} = \frac{1}{\sqrt{1 + (1/kr)^2}} = \frac{\vec{c}_{a_r}(r,\omega)}{c}
\]

The real and imaginary parts of the complex radial \textit{specific} acoustic impedance associated with the point monopole sound source are thus:

\[
z_{a_r}^r(r,\omega) = \text{Re}\left\{\vec{Z}_{a_r}(r,\omega)\right\} = z_o \frac{1}{1 + (1/kr)^2} \quad \text{and:} \quad z_{a_r}^i(r,\omega) = \text{Im}\left\{\vec{Z}_{a_r}(r,\omega)\right\} = z_o \frac{1/kr}{1 + (1/kr)^2}
\]
Likewise, the real and imaginary parts of the complex radial acoustic energy flow velocity are:

\[
c_i^r (r, \omega) = \text{Re}\{\tilde{c}_a (r, \omega)\} = c \frac{1}{1 + (1/kr)^2} \quad \text{and:} \quad c_i^r (r, \omega) = \text{Im}\{\tilde{c}_a (r, \omega)\} = c \frac{1/kr}{1 + (1/kr)^2}
\]

The phase of the complex radial specific acoustic impedance and complex radial acoustic energy flow velocity is:

\[
\phi_{c_i} = \phi_{c_o} = \Delta \phi_{p-u} = \phi_p - \phi_u = \tan^{-1}(1/kr)
\]

(a) **In the “near” zone:** \(kr \ll 1\), the complex radial specific acoustic impedance and the complex radial energy flow velocity are both largely *imaginary* (i.e. *reactive*):

\[
\tilde{z}_a (r)/z_o = \tilde{c}_a (r)/c = [1 + i/kr]\sqrt{1 + (1/kr)^2} \sim i \cdot kr, \quad \text{increasing} \sim \text{linearly with} \ r,
\]

fractional magnitude \(\left|\tilde{z}_a (r)/z_o\right| = \left|\tilde{c}_a (r)/c\right| = 1/\sqrt{1 + (1/kr)^2} \sim kr \ll 1\) and phase \(\phi_{c_i} = \phi_{c_o} = \phi_p - \phi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(\infty) \sim 90^\circ\). Again, the complex sound field \(\tilde{S}(\hat{r}, t)\) in this region is *inertia*-like (i.e. *mass*-like), because \(z_{a_i}^r (r) = \text{Im}\{\tilde{z}_a (r)\} > 0\) for \(kr \ll 1\).

(b) **In the “intermediate zone:** \(kr \sim 1\), the complex radial specific acoustic impedance and the complex radial energy flow are both \sim an equal mix of *active* (i.e. *real*) and *reactive* (i.e. *imaginary*) components:

\[
\tilde{z}_a (r)/z_o = \tilde{c}_a (r)/c = [1 + i/kr]\sqrt{1 + (1/kr)^2} \sim (1+i)/2, \quad \text{with}
\]

fractional magnitude \(\left|\tilde{z}_a (r)/z_o\right| = \left|\tilde{c}_a (r)/c\right| = 1/\sqrt{1 + (1/kr)^2} \sim 1/\sqrt{2}\) and phase \(\phi_{c_i} = \phi_{c_o} = \phi_p - \phi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(1) \sim 45^\circ\).

(c) **In the “far” (or radiation) zone:** \(kr \gg 1\), the complex radial specific acoustic impedance and the complex radial energy flow velocity are largely *real* (i.e. *active*):

\[
\tilde{z}_a (r)/z_o = \tilde{c}_a (r)/c = 1/\sqrt{1 + (1/kr)^2} \sim 1 \quad \text{with magnitude}
\]

\[
\left|\tilde{z}_a (r)/z_o\right| = \left|\tilde{c}_a (r)/c\right| = 1/\sqrt{1 + (1/kr)^2} \sim 1\) and phase \(\phi_{c_i} = \phi_{c_o} = \phi_p - \phi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(0) \sim 0^\circ\).

The **frequency domain** complex radial acoustic intensity associated with the point acoustic monopole also points outward in the radial \(\hat{r}\) direction; it and its magnitude are:

\[
\tilde{I}_a (r, \omega) = \frac{1}{2} \tilde{p}(r, \omega) \cdot \tilde{u}_a^r (r, \omega) = \frac{1}{2} \frac{B_0^2}{z_o} \frac{1 + i/kr}{r^3} \quad \text{and:} \quad |\tilde{I}_a (r, \omega)| = \frac{1}{2} \frac{B_0^2}{z_o} \frac{1}{r^3} \sqrt{1 + (1/kr)^2}
\]

The real and imaginary parts of the **frequency domain** radial complex acoustic intensity are:

\[
I_i^r (r, \omega) = \text{Re}\{\tilde{I}_a (r, \omega)\} = \frac{1}{2} \frac{B_0^2}{z_o} \frac{1}{r^3} \quad \text{and:} \quad I_o^r (r, \omega) = \text{Im}\{\tilde{I}_a (r, \omega)\} = \frac{1}{2} \frac{B_0^2}{z_o} \frac{1}{kr}
\]
The phase of the **frequency domain** radial complex acoustic intensity is:

\[ \varphi_{ia} = \varphi_{za} = \varphi_{ca} = \Delta \varphi_{p-u} = \varphi_p - \varphi_u = \tan^{-1}(1/kr) \]

(a) **In the “near” zone**: \( kr \ll 1 \), the **frequency domain** complex radial acoustic intensity is largely **imaginary** (i.e. **reactive**):

\[ \vec{I}_{ia} (r, \omega) \sim i \frac{B_o^2}{2z_o kr^3} \], with magnitude \( |\vec{I}_{ia} (r, \omega)| \sim \frac{B_o^2}{2z_o kr^3} \) (decreasing as \( \sim 1/r^3 \)) and phase \( \varphi_{ia} = \varphi_{za} = \varphi_{ca} = \varphi_p - \varphi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(\infty) \sim 90^\circ \). Again, the complex sound field \( \vec{S}(\vec{r}, t) \) in this region is **inertia**-like (i.e. **mass**-like), because

\[ I_{ia}^i (r) = \text{Im}\{\vec{I}_{ia} (r)\} > 0 \text{ for } kr \ll 1. \]

(b) **In the “intermediate zone**: \( kr \sim 1 \), the **frequency domain** complex radial acoustic intensity is \( \sim \) an equal mix of **active** (i.e. **real**) and **reactive** (i.e. **imaginary**) components:

\[ \vec{I}_{ia} (r, \omega) \sim (1 + i) \frac{B_o^2}{4z_o r^2} \], with magnitude \( |\vec{I}_{ia} (r, \omega)| \sim \frac{B_o^2}{4z_o r^2} \) (decreasing slightly faster than \( \sim 1/r^2 \)) and phase \( \varphi_{ia} = \varphi_{za} = \varphi_{ca} = \varphi_p - \varphi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(1) \sim 45^\circ \).

(c) **In the “far” (or radiation) zone**: \( kr \gg 1 \), the **frequency domain** complex radial acoustic intensity is largely **real** (i.e. **active**):

\[ |\vec{I}_{ia} (r, \omega)| \sim \frac{B_o^2}{2z_o r^2} \] (decreasing as \( \sim 1/r^2 \)) and phase \( \varphi_{ia} = \varphi_{za} = \varphi_{ca} = \varphi_p - \varphi_u = \tan^{-1}(1/kr) \sim \tan^{-1}(0) \sim 0^\circ \).

The **frequency domain** complex acoustic power and its magnitude associated with the point monopole sound source are:

\[ \vec{P}_a (r, \omega) = \int_S \vec{I}_{ia} (r, \omega) \hat{r} \cdot d\vec{S} = 2\pi \frac{B_o^2}{z_o} \left[ 1 + \frac{i}{kr} \right] \text{ and: } |\vec{P}_a (r, \omega)| = 2\pi \frac{B_o^2}{z_o} \sqrt{1 + \left( \frac{1}{kr} \right)^2} \]

Notice that both the **frequency domain** point monopole complex acoustic power and its magnitude do indeed have an explicit \( r \)-dependence associated with them – we are used to thinking/told that they are **not** supposed to have an explicit \( r \)-dependence, because this is **only** true for the propagating portion of the complex acoustic power – i.e. the **real** acoustic power!!!

The real and imaginary parts of the **frequency domain** complex acoustic power are:

\[ P_a^r (r, \omega) = \text{Re}\{\vec{P}_a (r, \omega)\} = 2\pi \frac{B_o^2}{z_o} = \text{constant} \neq \text{fcn}(r) \]

and:

\[ P_a^i (r, \omega) = \text{Im}\{\vec{P}_a (r, \omega)\} = 2\pi \frac{B_o^2}{z_o} \frac{1}{kr} \Leftrightarrow \text{explicit fcn}(r)!!! \]

The phase associated with the **frequency domain** complex acoustic power is:

\[ \varphi_{pa} = \varphi_{ia} = \varphi_{za} = \varphi_{ca} = \Delta \varphi_{p-u} = \varphi_p - \varphi_u = \tan^{-1}(1/kr) \]
In our previous discussions of acoustic power $P_a$ associated e.g. with “point” sound sources, the context was always with regard to propagating sound – i.e. sound radiation – purely real (and/or time-averaged) acoustic power.

We see from above that the real component of the complex acoustic power of a monopole sound source – which is associated with propagating sound energy – is indeed a constant, whereas the imaginary component of the complex acoustic power of a monopole sound source – which is associated with non-propagating acoustic energy – is explicitly $r$-dependent, and especially so in the “near” zone, when $kr \ll 1$.

The purely real, scalar frequency domain potential, kinetic and total acoustic energy densities associated with a “point” monopole sound source radiating monochromatic, radially-outgoing spherical waves are:

$$w_{a\text{pot}}(r, \omega) \equiv \frac{1}{4} \left| \bar{\rho}(r, \omega) \right|^2 = \frac{1}{4} \frac{\rho_o c^2}{r^2} \frac{B_o^2}{4\pi^2} \quad (\text{Joules/m}^3)$$

$$w_{a\text{kin}}(r, \omega) \equiv \frac{1}{4} \rho_o \left( \bar{\ddot{u}}_r(r, \omega) \cdot \dddot{u}_r^*(r, \omega) \right) = \frac{1}{4} \frac{\rho_o}{r^2} \left| \ddot{u}_r(r, \omega) \right|^2 = \frac{\rho_o B_o^2}{4\pi^2 r^2} \left[ 1 + \left( \frac{1}{kr} \right)^2 \right] \quad (\text{Joules/m}^3)$$

$$w_{a\text{tot}}(r, \omega) \equiv w_{a\text{pot}}(r, \omega) + w_{a\text{kin}}(r, \omega) = \frac{\rho_o B_o^2}{4\pi^2 r^2} \left[ 1 + \left( \frac{1}{kr} \right)^2 \right] = \frac{\rho_o B_o^2}{4\pi^2 r^2} \left[ 2 + \left( \frac{1}{kr} \right)^2 \right] \quad (\text{Joules/m}^3)$$

Note that the ratio of potential energy density to kinetic energy density is:

$$\frac{w_{a\text{pot}}(r, \omega)}{w_{a\text{kin}}(r, \omega)} = \frac{1}{4} \frac{\left| \bar{\rho}(r, \omega) \right|^2}{\rho_o \left| \ddot{u}_r(r, \omega) \right|^2} = \frac{\frac{\rho_o B_o^2}{4\pi^2 r^2} \left[ 1 + \left( \frac{1}{kr} \right)^2 \right]}{\frac{\rho_o B_o^2}{4\pi^2 r^2}} = \frac{1}{\left[ 1 + \left( \frac{1}{kr} \right)^2 \right]} \leq 1$$

In the “near” zone, $kr \ll 1$ where the complex radial particle velocity, specific acoustic impedance, energy flow velocity, acoustic intensity and power are largely reactive (i.e. imaginary), $w_{a\text{pot}}(r, \omega) \ll w_{a\text{kin}}(r, \omega)$. Only in the “far” (i.e. radiation) zone, when $kr \gg 1$, and moreover, when $kr \to \infty$ (i.e. in free-field conditions) does $w_{a\text{pot}}(r, \omega) \approx w_{a\text{kin}}(r, \omega)$.

Please see/look at plots of the above complex acoustic quantities associated with the point acoustic monopole, available on the Physics 406 Software web-page, at the following URL:

http://online.physics.uiuc.edu/courses/phys406/406pom_sw.html
Example # 4: The Physical “Point” Monopole Sound Source:

At very low frequencies, a loudspeaker mounted in a fully-enclosed/sealed cabinet of characteristic dimension \(a\), with \(ka \ll 1\) (i.e. \(f \ll c/2\pi a\) \{using \(k \equiv 2\pi/\lambda\), \(\omega \equiv 2\pi f\) and \(c = f\lambda = \omega k\)\) approximates a “point” monopole sound source – the directivity factor, \(Q\) of a typical enclosed loudspeaker is very nearly 1 (i.e. isotropic) at frequencies \(f \ll c/2\pi a\). For example, for a typical “bookshelf”-type loudspeaker with \(a \sim 1\) ft \(\sim 0.3\) m, then \(f \ll c/2\pi a = 343/0.6\pi \sim 180\) Hz, or equivalently \(\lambda \gg 1.9\) m.

If we use the “spherical cow” approximation, i.e. model a physical monopole sound source as a radially-pulsating sphere of radius \(a\), subject to the restriction \(ka \ll 1\), then the acoustical properties of such a device (for \(r > a\)) will closely approximate that of an ideal, point monopole sound source.

How do we characterize the strength of a physical monopole sound source – i.e. a radially-pulsating sphere of radius \(a\)? Typically, this is done by considering the \{complex\} volumetric velocity (aka \(volume\) \(velocity\)) of the physical monopole source, evaluated at the radius \(a\) of the sphere – the \{radial\} outward volume rate (or flow) of fluid (i.e. air) from this sphere:

\[
\tilde{Q}_a \cdot e^{i\omega t} = \int S \tilde{u}_t (r = a, t) \hat{r} \cdot \hat{n} dS = 4\pi a \frac{\tilde{B}}{z_o} \left[1 - \frac{i}{ka}\right] e^{-ika} \cdot e^{i\omega t} \left( m^3 / s \right)
\]

Thus, the \{complex\} source strength/volume velocity of a physical monopole is:

\[
\tilde{Q}_a = 4\pi a \frac{\tilde{B}}{z_o} \left[1 - \frac{i}{ka}\right] e^{-ika} \left( m^3 / s \right)
\]

Since \(ka \ll 1\), then \(1/ka \gg 1\), we can therefore approximate this expression as:

\[
\tilde{Q}_a = -4\pi i \cdot \frac{\tilde{B}}{z_o} \left(\cos ka - i \sin ka\right) = -4\pi i \cdot \frac{\tilde{B}}{z_o} = -4\pi i \cdot \frac{\tilde{B}}{\rho_o c k} = -4\pi i \cdot \frac{\tilde{B}}{\rho_o \omega} \left( m^3 / s \right)
\]

Thus, we see that: \(\tilde{B} = i \frac{\rho_o \omega}{4\pi} \tilde{Q}_a\) \((Pascal - m = kg / s^2)\)

Expressed in terms of the complex source strength/volume velocity \(\tilde{Q}_a\) \((m^3 / s)\) of the physical monopole sphere of radius \(a\), the time-domain and frequency-domain complex pressure and radial particle velocity associated with this sound source are (for \(r > a\)):

\[
\tilde{p}(r, t) = \frac{\tilde{B}}{r} e^{i(\omega t - kr)} = i \frac{\omega}{4\pi} \frac{\tilde{Q}_a}{r} e^{i(\omega t - kr)} = \check{p}(r, \omega) e^{i\omega t} \quad \text{i.e.} \quad \check{p}(r, \omega) = i \frac{\omega}{4\pi c} \frac{\tilde{Q}_a}{r} e^{-ikr}
\]

\[
\tilde{u}_t(r, t) = \frac{1}{z_o} \frac{\tilde{B}}{r} \left[1 - \frac{i}{kr}\right] e^{i(\omega t - kr)} = i \frac{\omega}{4\pi c} \frac{\tilde{Q}_a}{r} \left[1 - \frac{i}{kr}\right] e^{i(\omega t - kr)} = \check{u}_t(r, \omega) e^{i\omega t} \quad \text{i.e.} \quad \check{u}_t(r, \omega) = i \frac{\omega}{4\pi c} \frac{\tilde{Q}_a}{r} \left[1 - \frac{i}{kr}\right] e^{-ikr}
\]
The complex radial specific acoustic impedance and energy flow velocity are (unchanged):

\[
\frac{\tilde{z}_{\text{a},r}(r,\omega)}{z_o} = \frac{1}{\left[1 - i/kr\right]} = \frac{\left[1 + i/kr\right]}{\left[1 + \left(1/kr\right)^2\right]} = \frac{\tilde{c}_{\text{a},r}(r,\omega)}{c}
\]

The frequency-domain complex radial acoustic intensity (for \(r > a\)) is:

\[
\tilde{I}_{\text{a},r}(r,\omega) = \frac{1}{2} \tilde{p}(r,\omega) \tilde{u}^*_r(r,\omega) = \frac{1}{2} \frac{1}{z_o} \left| \frac{\tilde{B}}{r^2} \right|^2 \left[1 + i \frac{1}{kr}\right] = \frac{\rho_o \omega^2}{32\pi^2 c} \frac{\tilde{Q}}{r^2} \left[1 + i \frac{1}{kr}\right]
\]

Note that \(\tilde{I}_{\text{a},r}(r,\omega) \propto f^2\) - i.e. an acoustic monopole sound source is not efficient at very low frequencies in terms of generating sound….

The frequency-domain complex acoustic power associated with the physical monopole sound source (for \(r > a\)) is:

\[
\tilde{P}_o(r,\omega) = \int_S \tilde{I}_{\text{a},r}(r,\omega) \hat{r} \cdot d\vec{S} = 2\pi \frac{1}{z_o} \left| \frac{\tilde{B}}{r^2} \right|^2 \left[1 + i \frac{1}{kr}\right] = \rho_o \omega^2 \frac{\tilde{Q}}{8\pi c} \left[1 + i \frac{1}{kr}\right]
\]

The frequency-domain potential, kinetic and total energy densities associated with the physical monopole sound source (for \(r > a\)) are:

\[
w_{\text{a},\text{pot}}(r,\omega) = \frac{1}{4} \frac{\left| \tilde{p}(r,\omega) \right|^2}{\rho_o c^2} = \frac{\rho_o \omega^2}{64\pi^2 c^2} \frac{\left| \tilde{Q} \right|^2}{r^2} \text{ (Joules/m}^3\text{)}
\]

\[
w_{\text{a},\text{kin}}(r,\omega) = \frac{1}{4} \rho_o \left| \tilde{u}_r(r,\omega) \cdot \tilde{u}^*_r(r,\omega) \right| = \frac{1}{4} \rho_o \left| \tilde{u}_r(r,\omega) \right|^2 = \frac{\rho_o \omega^2}{64\pi^2 c^2} \frac{\left| \tilde{Q} \right|^2}{r^2} \left[1 + \left(\frac{1}{kr}\right)^2\right] \text{ (Joules/m}^3\text{)}
\]

\[
w_{\text{a},\text{tot}}(r,\omega) = w_{\text{a},\text{pot}}(r,\omega) + w_{\text{a},\text{kin}}(r,\omega) = \frac{\rho_o \omega^2}{64\pi^2 c^2} \frac{\tilde{Q}^2}{r^2} \left[2 + \left(\frac{1}{kr}\right)^2\right] \text{ (Joules/m}^3\text{)}
\]
Example #5: The Compact/Physical Dipole Sound Source:

By the principle of linear superposition {for SPL’s ≪ 134 dB}, we can create a so-called compact/physical dipole sound source using two out-of-phase physical monopole sources, of source strength/volume velocity ±Q, and separated from each other by a distance 2d, and subject to the requirement that kd ≪ 1 (i.e. f ≪ c/2πd or d ≪ c/2πf), as shown in the figure below:

The time-domain and the frequency-domain total/resultant complex over-pressure amplitudes at the observer/listener’s point P(\vec{r}) in the above figure is the linear sum of the individual complex over-pressures associated with each monopole source:

\[ \tilde{p}_{\text{tot}}(\vec{r}, t) = \tilde{p}_1(\vec{r}, t) + \tilde{p}_2(\vec{r}, t) = \tilde{B} \left[ \frac{1}{r_1} e^{-ik_1} - \frac{1}{r_2} e^{-ik_2} \right] e^{i\omega t} = i \frac{P_o \alpha \Omega}{4\pi} \left[ \frac{1}{r_1} e^{-ik_1} - \frac{1}{r_2} e^{-ik_2} \right] e^{i\omega t} = \tilde{p}_{\text{tot}}(\vec{r}, \omega) e^{i\omega t} \]

The time-domain and frequency-domain total/resultant complex particle velocity at the observation/listener’s point P(\vec{r}) in the above figure is the vector sum of the individual complex particle velocities associated with each monopole source:

\[ \tilde{u}_{\text{tot}}(\vec{r}, t) = \tilde{u}_1(\vec{r}, t) + \tilde{u}_2(\vec{r}, t) = \tilde{u}_1(\vec{r}, t) \hat{\vec{r}}_1 + \tilde{u}_2(\vec{r}, t) \hat{\vec{r}}_2 \]
The acoustic monopole sources, of source strength/volume velocity \( \pm \tilde{Q}_a \) are located at \( \tilde{d}_1 = +d \hat{z} \) and \( \tilde{d}_2 = -d \hat{z} \) with \( |\tilde{d}_1| = |\tilde{d}_2| = d \). Vectorially, we see that \( \tilde{r} = \tilde{d}_1 + \tilde{r}_1 \) and also that \( \tilde{r} = \tilde{d}_2 + \tilde{r}_2 \), with \( \tilde{r} = r \hat{r} \) and \( |\tilde{r}| = r = \sqrt{x^2 + y^2 + z^2} \). In Cartesian coordinates \( \hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \).

We also see that: \( \tilde{r}_1 = \tilde{r} - \tilde{d}_1 \) and \( \tilde{r}_2 = \tilde{r} - \tilde{d}_2 \). Using the law of cosines \( c^2 = a^2 + b^2 - 2ab \cos \theta \):

\[
|\tilde{r}_1| = r_1 = \sqrt{r^2 + d^2 - 2rd \cos \theta} \quad \text{and} \quad |\tilde{r}_2| = r_2 = \sqrt{r^2 + d^2 - 2rd \cos (\pi - \theta)} = \sqrt{r^2 + d^2 + 2rd \cos \theta} ,
\]

with \( \tilde{r}_1 = |\tilde{r}_1| \hat{r}_1 = \tilde{r} - \tilde{d}_1 = \tilde{r} - d \hat{z} \) and \( \tilde{r}_2 = |\tilde{r}_2| \hat{r}_2 = \tilde{r} - \tilde{d}_2 = \tilde{r} + d \hat{z} \) with \( \hat{r}_1 = (r \hat{r} - d \hat{z})/r_1 \) and \( \hat{r}_2 = (r \hat{r} + d \hat{z})/r_2 \).

Since \( \tilde{r}_1 \neq \tilde{r}_2 \neq \tilde{r} \) (especially in the “near-field” region), the above expression for the total / resultant complex vector particle velocity \( \tilde{u}_{tot}(\tilde{r},t) \) is not easy to evaluate, analytically. However, note that it does simplify considerably in the “far”-field region, where \( \tilde{r}_1 = \tilde{r}_2 = \tilde{r} \) and \( r_1 = r_2 = r \gg d \).

It is quite clear from the above formulae for \( \tilde{p}_{tot}(\tilde{r},t) \) and \( \tilde{u}_{tot}(\tilde{r},t) \) {as well as from previous P406 lectures on sound interference effects with 2 (or more) sources} that interference effects will indeed manifest themselves here in this situation, albeit in a much more complicated manner….

However, the nature of this problem is such that all of the above quantities that we have calculated analytically e.g. for the various simpler sound sources can be also easily coded up on a computer, e.g. using MATLAB, Mathematica or e.g. a C/C++ based-program coupled to a graphics software package for plots, not just for them, but for this problem as well…

Computational calculations can also be done for {arbitrarily} higher-order acoustic multipoles – e.g. linear {and/or crossed (aka lateral)} quadrupoles (the tuning fork is an example of a linear quadrupole), sextupoles, octupoles, hexadecapoles, arbitrary linear 1-D acoustic arrays, 2-D/3-D acoustic arrays, all using the principle of linear superposition for \( N \) monopole sound sources… an arbitrary sound source can always be decomposed into a linear combination of acoustic multipoles, with suitably chosen complex coefficients – the multipole strengths…

Some very nice animation demos of pressure fields for monopole, dipole, quadrupole… sound sources exist e.g. at Prof. Dan Russell’s website:

http://www.acs.psu.edu/drussell/Demos/rad2/mdq.html

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Example # 6: The Plane Circular Piston (A Simple Acoustics Model of a Loudspeaker):

Consider a longitudinally oscillating piston of radius \( a \) mounted on an infinite, perfectly rigid \( \{ i.e. \) immovable \} planar baffle \( \{ \text{oriented in the } x\text{-}y \text{ plane at } z = 0 \} \) as shown in the figure below.

The surface of the plane circular piston oscillates harmonically back and forth along the \( \hat{z} \) -axis with complex velocity \( \vec{u}_p(z,t) = u_0 e^{i\omega t} \hat{z} \). What is the resulting complex pressure \( \vec{p}(\vec{r},t) = \vec{p}(r,\theta,\varphi = 0,t) \) \( \text{e.g. at an observation/listener point } \vec{r} = (r,\theta,\varphi = 0) \text{ lying in the } x\text{-}z \text{ plane?} \)

Note that the harmonically oscillating baffled plane circular piston is a \textit{spatially-extended} sound source, \( \text{i.e. it is not a point sound source. We can consider the harmonically oscillating baffled plane circular piston as a coherent linear superposition of infinitesimal, point sound sources, each of which is \textit{isotropically} radiate sound waves into the forward hemisphere \( \{ \hat{z} > 0 \} \) \{since the piston is mounted on an infinite, rigid baffle lying in the } x\text{-}y \text{ plane at } z = 0 \}. \) Thus, each \textit{infinitesimal} area element \( dS \) on the plane circular piston acts like a \textit{point} sound source, with infinitesimal \textit{volume velocity strength} \( dQ = u_0 \, dS \ (m^3/s) \).

From \textbf{Example # 4} above, we learned that the complex \textit{time-domain} over-pressure amplitude associated with an \textit{unbaffled physical} point sound source \( \{ \text{located at the origin} \} \) of volume velocity strength \( \vec{Q}_a \) harmonically radiating sound waves isotropically into \( 4\pi \) steradians (with \textit{directivity} \( Q = 1 \) ) a distance \( r \) away from this sound source is:

\[
\vec{p}_{\text{un-baff}}(r,t) = \frac{\vec{B}}{r} e^{i(\omega t - kr)} = i \frac{\rho_0 c_0 Q_a}{4\pi} \frac{u_0 e^{i\omega t}}{r} e^{i(\omega t - kr)}
\]

However, for a \textit{baffled} physical point sound source \( \{ \text{located at the origin} \} \) of volume velocity strength \( \vec{Q}_a \) harmonically radiating sound waves isotropically \textit{only} into the \( 2\pi \) steradians of the \( \{ \hat{z} > 0 \} \) forward hemisphere (with \textit{directivity} \( Q = 2 \) ) the complex \textit{time-domain} over-pressure at a distance \( r \) away from this sound source is \( 2 \times \text{this, i.e.:} \)

\[
\vec{p}_{\text{baff}}(r,t) = \frac{\vec{B}}{r} e^{i(\omega t - kr)} = i \frac{\rho_0 c_0 Q_a}{4\pi} \frac{u_0 e^{i\omega t}}{r} e^{i(\omega t - kr)}
\]
\[ \bar{p}_{\text{bafl}}(r, t) = 2 \bar{p}_{\text{at-bafl}}(r, t) = 2 \frac{B}{r} e^{i(kz - \omega t)} = 2i \rho_o \omega \frac{\bar{Q}}{4\pi r} e^{i(kz - \omega t)} = i \rho_o \omega \frac{\bar{Q}}{2\pi r} e^{i(kz - \omega t)} = i \rho_o (r, \omega) e^{i(kz - \omega t)} \]

The **amplitude** of the complex over-pressure a distance \( r \) away from the **baffled** point sound source {located at the origin} is: \( \bar{p}_o(r, \omega) = \rho_o \omega \bar{Q}_o / 2\pi r \).

Hence, the **amplitude** of the complex over-pressure a distance \( r' \) away from the **baffled infinitesimal** point sound source associated with the infinitesimal area element \( dS \) located at the point \( \bar{\rho} \) on the surface of the plane circular piston {see figure above} is: \( \bar{p}_o(r', \omega) = \rho_o \omega \bar{Q}_o / 2\pi r' \).

Hence the **infinitesimal contribution** to the complex **time-domain** over-pressure amplitude a distance \( r' \) away from the **baffled infinitesimal** point sound source associated with the infinitesimal area element \( dS \) located at the point \( \bar{\rho} \) on the surface of the plane circular piston is: \( dp(r', \omega) = \rho_o \omega dQ / 2\pi r' = \rho_o \omega u_o dS / 2\pi r' \) where in the last step, we used the infinitesimal relation \( dQ = u_o dS \).

The corresponding **infinitesimal contribution** to the complex **time-domain** over-pressure is:

\[ dp(r', t) = dp(r', t) e^{i(kz - \omega t)} = i \rho_o \omega u_o \frac{dQ}{r'} e^{i(kz - \omega t)} = i \rho_o \omega u_o \frac{dS}{r'} e^{i(kz - \omega t)} dS \]

Then, we simply need to **sum up** all of the individual infinitesimal **contribution(s)** \( dp(r', t) \) – *i.e.* we need to **integrate** \( dp(r', t) \) over the area of the plane circular piston, in order to obtain the **total** complex **time-domain** over-pressure **amplitude** a distance \( r \) away from the **center** of the **baffled** harmonically oscillating plane circular piston, {which is taken to be the local origin of coordinates – see above figure}:

\[ \bar{p}(r, t) = \int dp(r', t) = \int i \rho_o \omega u_o \frac{dQ}{2\pi r'} e^{i(kz - \omega t)} dS = i \rho_o \omega u_o \int \frac{1}{r'} e^{i(kz - \omega t)} dS \]

Referring to the above figure, we need to express \( r' \) in terms of \( r \), \( \rho \) and angles \( \theta \) and \( \varphi \). Vectorially \( \bar{r} = \bar{\rho} + \bar{r}' \), hence \( \bar{r}' = \bar{r} - \bar{\rho} \), then:

\[ r'^2 = |\bar{r}'|^2 = \bar{r}' \cdot \bar{r}' = (\bar{r} - \bar{\rho}) \cdot (\bar{r} - \bar{\rho}) = \bar{r} \cdot \bar{r} - 2 \bar{\rho} \cdot \bar{r} + \bar{\rho} \cdot \bar{\rho} = |\bar{r}|^2 - 2 \bar{\rho} \cdot \bar{r} + |\bar{\rho}|^2 = r^2 - 2 \bar{\rho} \cdot \bar{r} + \rho^2 \]

The vector \( \bar{\rho} \) lies in the \( x-y \) plane, and has components: \( \bar{\rho} = \rho \hat{x} + \rho \hat{y} = \rho \cos \varphi \hat{x} + \rho \sin \varphi \hat{y} \).

It has **magnitude**: \( \rho = |\bar{\rho}| = \sqrt{\rho_x^2 + \rho_y^2} = \sqrt{\rho^2 \cos^2 \varphi + \rho^2 \sin^2 \varphi} = \rho \sqrt{\cos^2 \varphi + \sin^2 \varphi} = \rho \).

The vector \( \bar{r} \) \{n.b. fixed in space\} has components \{for the observation/listener position at \( \bar{r} \) lying **off-axis**, somewhere in the \( x-z \) plane\}: \( \bar{r} = r \hat{x} + 0 + r \hat{z} = x \hat{x} + z \hat{z} = r \sin \theta \hat{x} + r \cos \theta \hat{z} \) and has **magnitude**: \( r = |\bar{r}| = \sqrt{r_x^2 + r_z^2} = \sqrt{x^2 + z^2} = \sqrt{r^2 \sin^2 \theta + r^2 \cos^2 \theta} = r \sqrt{\sin^2 \theta + \cos^2 \theta} = r \)

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Thus, the dot product
\[ \vec{\rho} \cdot \vec{r} = (\rho \cos \phi \hat{x} + \rho \sin \phi \hat{y}) \cdot (r \sin \theta \hat{x} + r \cos \theta \hat{z}) = \rho r \sin \theta \cos \phi . \]

Hence:
\[ r'^2 = r^2 - 2\vec{\rho} \cdot \vec{r} + \rho^2 = r^2 - 2\rho r \sin \theta \cos \phi + \rho^2 , \]

Or:
\[ r' = \sqrt{r'^2} = \sqrt{r^2 - 2\rho r \sin \theta \cos \phi + \rho^2} \]

Next, if we carry out the integration of the above integral over the surface area of the plane circular piston of radius \( a \) using \( \) e.g. \ the method of annular strips, the area element can be written as \( dS = \rho d\phi \cdot d\rho \). The surface integral becomes:

\[
\tilde{p}(r, t) = i \frac{\rho_o \omega_o \rho}{2\pi} \int_0^{2\pi} \int_0^{\rho_o} \frac{1}{r'} e^{i(\omega - kr')} \rho d\phi d\rho
\]

Or:
\[
= i \frac{\rho_o \omega_o \rho}{2\pi} e^{i\omega t} \int_0^{\rho_o} \frac{1}{\sqrt{r^2 - 2\rho r \sin \theta \cos \phi + \rho^2}} e^{-i k \sqrt{r^2 - 2\rho r \sin \theta \cos \phi + \rho^2}} \rho d\phi d\rho
\]

This “off-axis” integral is not easy to carry out – analytically – it is a so-called elliptic integral (of the “würst” kind). It can be done e.g. numerically, on a computer. If the observer/listener position is located somewhere on the \( \hat{z} \)-axis, i.e. \( \vec{r} = z \hat{z} \) \{when \( \theta = 0 \}\), the “on-axis” integral is much easier to carry out – analytically, because the dot product \( \vec{\rho} \cdot \vec{r} \) vanishes, the “on-axis” problem has axial symmetry \{i.e. has rotational invariance about the \( \hat{z} \)-axis\}:

\[
\tilde{p}(r = z, t) = i \frac{\rho_o \omega_o \rho}{2\pi} e^{i\omega t} \int_0^{\rho_o} \frac{\rho}{\sqrt{z^2 + \rho^2}} e^{-i k \sqrt{z^2 + \rho^2}} d\rho
\]

Noting also that \( c = \omega/k \), \( z_o \equiv \rho_o c \) and \( \rho_o = z_o u_o = \rho_o c u_o \), the on-axis time-domain complex over-pressure is thus:

\[
\tilde{p}(r = z, t) = -i \frac{\rho_o \omega_o \rho}{\rho_c} e^{i\omega t} \int_0^{\rho_o} \left[ \frac{e^{-i k \sqrt{z^2 + \rho^2}}}{ik} \right] d\rho = -i \rho_o \omega_o \rho u_o e^{i\omega t} \int_0^{\rho_o} \left[ \frac{e^{-i k \sqrt{z^2 + \rho^2}}}{ik} \right] d\rho
\]

\[
= -i \frac{\kappa \rho_o \omega_o \rho}{\kappa k} e^{i\omega t} \int_0^{\rho_o} \left[ \frac{e^{-i k \sqrt{z^2 + \rho^2}}}{ik} \right] d\rho = \rho_o u_o \left[ 1 - e^{-i k \sqrt{z^2 + \rho^2}} \right] \cdot e^{i(\omega - k\rho)}
\]

\[ = p_o \left[ 1 - e^{-i k \sqrt{z^2 + \rho^2}} \right] \cdot e^{i(\omega - k\rho)} \cdot e^{i\omega t} = \tilde{p}(r = z, \omega) \cdot e^{i\omega t} \]
The real and imaginary components of the on-axis frequency-domain complex over-pressure are:

\[ p_r (r = z, \omega) = \text{Re} \{ \tilde{p}(r = z, \omega) \} = p_o \left[ \cos k z - \cos k z \sqrt{1 + (a/z)^2} \right] \]
\[ p_i (r = z, \omega) = \text{Im} \{ \tilde{p}(r = z, \omega) \} = -p_o \left[ \sin k z - \sin k z \sqrt{1 + (a/z)^2} \right] \]

The magnitude of the on-axis frequency-domain complex over-pressure amplitude is:

\[ |\tilde{p}(r = z)| = \sqrt{\tilde{p}(r = z) \tilde{p}^*(r = z) = \sqrt{2} p_o \left[ 1 - \cos k z \left( \sqrt{1 + (a/z)^2} - 1 \right) \right]} \]

Using the trigonometric identity \( \cos 2A = 1 - 2\sin^2 A \), this can be equivalently written as:

\[ |\tilde{p}(r = z)| = p_o \left| \frac{1}{2} k z \left( \sqrt{1 + (a/z)^2} - 1 \right) \right| \]

For \( z \gg a \), the Taylor series expansion for \( \sqrt{1 + (a/z)^2} = 1 + \frac{1}{2} (a/z)^2 \). and if additionally \( z \gg \frac{1}{2} k a^2 = \frac{1}{2} (2\pi/\lambda) a^2 = \pi a^2/\lambda \equiv \text{Rayleigh length} \), then the magnitude of the “far-field” on-axis frequency-domain complex over-pressure amplitude is:

\[ |\tilde{p}_{\text{far}}(r = z)| = \frac{1}{2} p_o \left( a/z \right) k a = p_o \left( \pi a^2/\lambda z \right) \]

Note that the magnitude of the “far-field” on-axis frequency-domain complex over-pressure is bounded by:

\[ 0 \leq |\tilde{p}_{\text{far}}(r = z)| \leq 2 p_o \quad \text{for} \quad 0 \leq r = z \leq \infty \]

Extrema of \( |\tilde{p}(r = z)| \) occur when: \( \frac{1}{2} k z \left( \sqrt{1 + (a/z)^2} - 1 \right) = \frac{1}{2} m \pi, \quad m = 0,1,2,3,4,5,... \) Solving the quadratic equation for \( z \), \( |\tilde{p}(r = z)| \) extrema occur when: \( z_m/a = (a/m\lambda) - (m\lambda/4a), \quad m = 1,2,3,4,5,... \)

Coming in from \( z = \infty \), the first maxima \( |\tilde{p}(r = z_i)|_{\text{max}} \) occurs at: \( z_i/a = (a/\lambda) - (\lambda/4a) \), the next maxima \( |\tilde{p}(r = z_3)|_{\text{max}} \) occurs at: \( z_3/a = (a/3\lambda) - (3\lambda/4a) \), and so on. Zeroes of \( |\tilde{p}(r = z_m)| \) occur in between the local maxima, when \( m = 2,4,6,8,... \)

Thus, having obtained the on-axis complex time-domain over-pressure amplitude \( \tilde{p}(r = z,t) \), we next use the {linearized} Euler’s equation for inviscid fluid flow to obtain the on-axis complex time-domain particle velocity \( \tilde{u}(r = z,\omega,t) \):

\[ \frac{\partial \tilde{u}(r = z,t)}{\partial t} = -\frac{1}{\rho_o} \hat{\nabla} \tilde{p}(r = z,t) = -\frac{1}{\rho_o} \frac{\partial \tilde{p}(r = z,t)}{\partial z} \hat{z} \]
Since the observer/listener position is on-axis at \( \hat{r} = \hat{z} = z \hat{z} \), and the on-axis complex time-domain over-pressure \( \tilde{p}(r = z, t) \) has axial symmetry (i.e. no \( \varphi \)-dependence), the gradient operator \( \tilde{V} = \partial / \partial r \hat{r} \Rightarrow \partial / \partial z \hat{z} \), and (after some derivative-taking and some algebra), the on-axis complex time-domain particle velocity is:

\[
\tilde{u}_z (r = z, t) = u_o \left[ 1 - e^{-ik \sqrt{z^2 + a^2 - z^2}} \right] - \left\{ \frac{z}{\sqrt{z^2 + a^2}} - 1 \right\} e^{-ik \sqrt{z^2 + a^2 - z^2}} e^{(\omega t - kz)}
\]

\[
= u_o \left[ 1 - \frac{1}{\sqrt{1 + (a/z)^2}} e^{-ikz \sqrt{1 + (a/z)^2}} \right] e^{(\omega t - kz)}
\]

\[
= u_o \left[ e^{-ikz} - \frac{1}{\sqrt{1 + (a/z)^2}} e^{-ikz \sqrt{1 + (a/z)^2}} \right] e^{i\omega t} = \tilde{u}_z (r = z, \omega) \cdot e^{i\omega t}
\]

The real and imaginary components of the on-axis frequency-domain complex particle velocity are:

\[
u_z (r = z, \omega) = \text{Re} \{ \tilde{u}_z (r = z, \omega) \} = u_o \left[ \cos kz - \frac{1}{\sqrt{1 + (a/z)^2}} \cos kz \sqrt{1 + (a/z)^2} \right]
\]

and:

\[
u_z (r = z, \omega) = \text{Im} \{ \tilde{u}_z (r = z, \omega) \} = -u_o \left[ \sin kz - \frac{1}{\sqrt{1 + (a/z)^2}} \sin kz \sqrt{1 + (a/z)^2} \right]
\]

The on-axis complex specific longitudinal acoustic impedance, using \( z_o \equiv \rho_o c = p_o / u_o \) is:

\[
\tilde{z}_o (r = z, \omega) = \frac{\tilde{p}(r = z, t)}{\tilde{u}_z (r = z, t)} = \frac{\tilde{p}(r = z, \omega) \cdot e^{i\omega t}}{\tilde{u}_z (r = z, \omega) \cdot e^{i\omega t}} = \tilde{p}(r = z, \omega)
\]

\[
p_o \left[ 1 - e^{-ikz \sqrt{1 + (a/z)^2}} \right] \left[ 1 - e^{-ikz \sqrt{1 + (a/z)^2}} \right] = z_o
\]

\[
u_o \left[ 1 - \frac{1}{\sqrt{1 + (a/z)^2}} e^{-ikz \sqrt{1 + (a/z)^2}} \right] \left[ 1 - \frac{1}{\sqrt{1 + (a/z)^2}} e^{-ikz \sqrt{1 + (a/z)^2}} \right] = z_o
\]
Using the relations $\tilde{z}_{a_z} = \rho_o \tilde{c}_{a_z}$ and $z_o = \rho_o c$ the above relation can be rewritten as a dimensionless quantity:

$$\frac{\tilde{z}_{a_z}(r = z, \omega)}{z_o} = \frac{1 - e^{-ikz\sqrt{1+(a/z)^2}}}{1 - \frac{1}{\sqrt{1+(a/z)^2}} e^{-i\omega z\sqrt{1+(a/z)^2}}} = \frac{\tilde{c}_{a_z}(r = z, \omega)}{c}$$

The frequency-domain on-axis complex longitudinal acoustic intensity is:

$$\tilde{I}_{a_z}(r = z, \omega) = \frac{1}{2} \bar{p}(r = z, \omega) \bar{u}_z^*(r = z, \omega)$$

$$= \frac{1}{2} \rho_o u_o \left[ 1 - e^{-i\omega z\sqrt{1+(a/z)^2}} \right] \left[ 1 - \frac{1}{\sqrt{1+(a/z)^2}} e^{+i\omega z\sqrt{1+(a/z)^2}} \right]$$

$$= \frac{1}{2} I_o \left[ 1 - e^{-i\omega z\sqrt{1+(a/z)^2}} \right] \left[ 1 - \frac{1}{\sqrt{1+(a/z)^2}} e^{+i\omega z\sqrt{1+(a/z)^2}} \right]$$

We leave it as an exercise for the interested/motivated reader to explicitly obtain the real and imaginary parts of on-axis $\tilde{z}_{a_z}(r = z, \omega)$ and $\tilde{I}_{a_z}(r = z, \omega)$, calculate the {purely real} on-axis potential/kinetic/total acoustic energy densities, etc.

Please see/look at plots of the above complex acoustic quantities associated with the baffled plane circular piston, available on the Physics 406 Software web-page, at the following URL:

http://courses.physics.illinois.edu/phys406/406pom_sw.html
Example # 7: The Uniform Planar Rigid-Walled 2-D Duct:

The final example we wish to discuss is that of sound propagation of monochromatic waves in an infinitely-long uniform planar duct consisting of two infinite, parallel, perfectly reflecting and rigid walls separated by a perpendicular distance \( a \), as shown in the figure below:

The wave equation for the complex scalar over-pressure field is:

\[
\nabla^2 \tilde{p}(\vec{r},t) - \frac{1}{c^2} \frac{\partial^2 \tilde{p}(\vec{r},t)}{\partial t^2} = 0
\]

The sound propagation direction is in the +\( \hat{x} \)-direction; note that the sound waves are not constrained in the \( z \)-direction (+\( \hat{z} \) points out of the page), whereas sound waves are constrained in the \( y \)-direction, being allowed only in the region between the two infinite, parallel walls: \( 0 \leq y \leq a \). Thus, this problem is only a 2-D problem in \((x,y)\) rectangular coordinates.

The gradient \( \tilde{\nabla} \) and Laplacian \( \nabla^2 \) operators in 2-D Cartesian/rectangular coordinates are:

\[
\tilde{\nabla} = \frac{\partial}{\partial x} \hat{x} + \frac{\partial}{\partial y} \hat{y}
\]
and:

\[
\nabla^2 \equiv \tilde{\nabla} \cdot \tilde{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
\]

The 2-D wave equation for the complex \textit{time-domain} scalar pressure field is thus:

\[
\frac{\partial^2 \tilde{p}(x,y,t)}{\partial x^2} + \frac{\partial^2 \tilde{p}(x,y,t)}{\partial y^2} - \frac{1}{c^2} \frac{\partial^2 \tilde{p}(x,y,t)}{\partial t^2} = 0
\]

We seek +\( \hat{x} \)-propagating wave product-type solutions of the general form:

\[
\tilde{p}(x,y,t) = \tilde{X}(x)\tilde{Y}(y)\tilde{T}(t) = Ae^{-ik_x x}e^{ik_y y}e^{i\omega t}
\]

The homogeneous wave equation is separable in these variables; the resulting \textit{characteristic equation} for the wavenumber \( k \) is: \( k^2 = k_x^2 + k_y^2 \), with accompanying \textit{dispersion relation} \( k^2 = \omega^2 / c^2 \).

{The details of this separation-of-variables technique for the 2-D wave equation are given in the P406 Lecture Notes on “Mathematical Musical Physics of the Wave Equation” p. 12-13}. 

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The boundary condition on the pressure at the two infinite, rigid parallel walls in the $\hat{y}$-direction is that there are pressure anti-nodes at $y = 0$ and $y = a$. Mathematically, this requires Neumann-type boundary conditions on the walls, i.e. $\partial \bar{p}(x, y = 0, t)/\partial y = \partial \bar{p}(x, y = a, t)/\partial y = 0$, requiring cosine-type solutions for $\bar{Y}(y) = e^{\pm ik_y y}$, i.e. $\bar{Y}(y) \sim e^{ik_y y} + e^{-ik_y y} \sim \cos k_y y$ such that:

$$\cos k_y y \bigg|_{y=0,a} = 1 \text{ with } \partial \cos k_y y / \partial y \bigg|_{y=0,a} = \sin k_y y \bigg|_{y=0,a} = 0 \implies k_y a = n\pi, \ n = 0, 1, 2, 3...$$

Thus we see that: $k_y a = n\pi$ or: $k_y = n\pi/a$, $n = 0, 1, 2, 3...$. Thus, for any given frequency $f = \omega/2\pi$, there are an infinite number of possible solutions (aka eigenmodes) for this wave equation, each one of the general form:

$$\tilde{p}_n(x, y, t) = \tilde{A}_n \cos(n \pi y / a) e^{i(n\pi y - k_x x)} \text{ where: } k_x = \sqrt{k^2 - k_y^2} \implies k_n = \sqrt{k^2 - (n\pi/a)^2}$$

The transverse pressure distribution $\sim \cos(n \pi y / a)$ for $0 \leq y \leq a$ is a characteristic of the wall geometry associated with this problem – i.e. a duct; and one which is caused by multiple, perfect (i.e lossless) reflections of the pressure waves off of the duct walls as they propagate in the $+\hat{x}$-direction. The integer $n$ denotes the {duct-} mode of propagation.

The $n = 0$ mode is known as the axial plane-wave eigenmode of propagation. The $n \geq 1$ modes are collectively known as transverse duct eigenmodes. At a given frequency $f$, if a specific duct eigenmode $n$ is excited, it may only propagate along the duct with a unique axial wavenumber given by $k_n = \sqrt{k^2 - (n\pi/a)^2}$.

Note that for each/every duct eigenmode $n$ of propagation, there is an (angular) frequency $\omega$ for which the axial wavenumber $k_n = \sqrt{k^2 - (n\pi/a)^2} = \sqrt{(\omega/c)^2 - (n\pi/a)^2} = 0$. The so-called cutoff frequency for the $n^{th}$ mode is: $\omega_n^{\text{cutoff}} = n\pi c / a$ or: $f_n^{\text{cutoff}} = nc/2a$. Below this cutoff frequency, the duct eigenmode $n$ cannot propagate – it becomes an evanescent mode because the axial eigen-wavenumber $k_n$ becomes purely imaginary for $f < f_n^{\text{cutoff}} = nc/2a$ – i.e. the duct eigenmode $n$ is exponentially attenuated by a factor of $e^{-k_x x}$ when $f < f_n^{\text{cutoff}} = nc/2a$.

A plot of the dispersion curves - axial wavenumber $k_x = k_n$ vs. angular frequency $\omega$ showing the effect of the cutoff frequency vs. mode number $n = 0, 1, 2, 3, \ldots$ is shown in the figure below.
For a given (angular) frequency \( \omega \), in general, the total/net complex pressure amplitude is a sum over all modes – the allowed/propagating individual complex pressure eigenmodes \( \frac{c_{n}}{c_{n}} \leq n \leq n_{\text{cutoff}} \), where \( n_{\text{cutoff}} \) is the highest eigenmode number \( n \) such that

\[
k_{x} = k_{n} = \sqrt{(\omega/c)^2 - (n_{\text{cutoff}} \pi/a)^2} > 0, \quad \text{i.e.} \quad n_{\text{cutoff}} = \text{int} \{ \omega a / \pi c \} \quad (= \text{floor} \{ \omega a / \pi c \}), \quad \text{and.}
\]

the individual non-propagating modes \( n > n_{\text{cutoff}} \):

\[
\tilde{p}_{\text{tot}}(x,y,t) = \sum_{n=0}^{\infty} \tilde{p}_{n}(x,y,t) = \sum_{n=0}^{\infty} \tilde{A}_{n} \cos(n \pi y/a) e^{i(\omega t-k_{x}x)}
\]

{n.b. Far from the sound source, this reduces to the sum over propagating modes \( n \leq n_{\text{cutoff}} \).}

The complex time-domain 2-D particle velocity \( \tilde{u}(\vec{r},t) \) associated with this problem is obtained via use of Euler’s equation for inviscid fluid flow. Since \( \tilde{p}(\vec{r},t) = \tilde{p}(x,y,t) \neq fcn(z) \), then \( \nabla \tilde{p}(\vec{r},t) = \nabla \tilde{p}(x,y,t) = (\partial \tilde{p}(x,y,t)/\partial x) \hat{x} + (\partial \tilde{p}(x,y,t)/\partial y) \hat{y} \), thus the particle velocity can only be in the \( x, y \) direction(s), i.e. \( \tilde{u}(\vec{r},t) = \tilde{u}(x,y,t) \neq fcn(z) \) since:

\[
\begin{align*}
\frac{\partial \tilde{u}(\vec{r},t)}{\partial t} &= -\frac{1}{\rho_o} \nabla \tilde{p}(\vec{r},t) \\
\frac{\partial \tilde{u}(x,y,t)}{\partial t} &= -\frac{1}{\rho_o} \left( \frac{\partial \tilde{p}(x,y,t)}{\partial x} \hat{x} + \frac{\partial \tilde{p}(x,y,t)}{\partial y} \hat{y} \right)
\end{align*}
\]

Euler’s equation holds for each/every duct eigenmode \( n \). With \( \tilde{p}_{n}(x,y,t) = \tilde{A}_{n} \cos(n \pi y/a) e^{i(\omega t-k_{x}x)} \), the general form of the complex time-domain 2-D particle velocity for the \( n^{\text{th}} \) duct eigenmode is thus:

\[
\tilde{u}_{n}(x,y,t) = \frac{1}{\omega \rho_o} \tilde{A}_{n} \left[ k_{n} \cos(n \pi y/a) \hat{x} - i(n \pi/a) \sin(n \pi y/a) \hat{y} \right] e^{i(\omega t-k_{x}x)}
\]

where: \( k_{x} = k_{n} = \sqrt{k_{x}^{2} - (n \pi/a)^2} = \sqrt{(\omega/c)^2 - (n \pi/a)^2} \)
For a given (angular) frequency $\omega$, for each of the *propagating* duct eigenmodes $n \leq n_{\text{cutoff}}$, the **axial** component of the particle velocity $\tilde{u}_n(x, y, t) = \frac{1}{\omega \rho_o} \tilde{A}_n k_n \cos(n\pi y/a) e^{i\omega (at-k_n x)} \hat{x}$ is in-phase with the complex pressure $\tilde{p}_n(x, y, t) = \tilde{A}_n \cos(n\pi y/a) e^{i\omega (at-k_n x)}$, whereas the **transverse** component of the particle velocity $\tilde{u}_n(x, y, t) = \frac{-i}{\omega \rho_o} \tilde{A}_n (n\pi/a) \sin(n\pi y/a) e^{i\omega (at-k_n x)} \hat{y}$ is in quadrature (*i.e.* $90^\circ$ out of phase) with the complex over-pressure amplitude.

The **total/net** complex time-domain 2-D particle velocity is likewise given by:

$$\tilde{u}(x, y, t) = \sum_{n=0}^{\infty} \tilde{u}_n(x, y, t) = \frac{1}{\omega \rho_o} \sum_{n=0}^{\infty} \tilde{A}_n \left[ k_n \cos(n\pi y/a) \hat{x} - i(n\pi/a) \sin(n\pi y/a) \hat{y} \right] e^{i\omega (at-k_n x)}$$

where: $k_x = k_n = \sqrt{k^2 - (n\pi/a)^2} = \sqrt{(\omega/c)^2 - (n\pi/a)^2}$ and: $n_{\text{cutoff}} = \text{int}\{\omega a/\pi c\} = \text{floor}\{\omega a/\pi c\}$.

The total/net complex pressure wave $\tilde{p}_{\text{tot}}(x, y, t) = \sum_{n=0}^{\infty} \tilde{p}_n(x, y, t)$ and 2-D particle velocity wave $\tilde{u}(x, y, t) = \sum_{n=0}^{\infty} \tilde{u}_n(x, y, t)$ that propagate in a duct depend on the details of the coupling of the sound source to that duct. For example, a 2-D “line” monopole source of volumetric velocity per unit length $Q'_n = Q_o/L \left( \text{m}^2/\text{s} \right)$ located *e.g.* at $(x, y) = (0, y_o)$ in the duct will produce modal pressure amplitudes of:

$$\tilde{A}_n = \frac{\omega \rho_o Q'_o}{k_n a} \cos(n\pi y_o/a)$$

This relation predicts that the $n^{th}$ modal pressure amplitude $\tilde{A}_n$ becomes *infinite* at the cutoff frequency for that mode, $\omega_n^{\text{cutoff}} = n\pi c/a$ when $k_x = k_n = \sqrt{\left(\omega_n^{\text{cutoff}} / c\right)^2 - (n\pi/a)^2} = 0$!! However, in the real world, nothing becomes infinite – *e.g.* the **finite** impedance of a real/physical acoustic source precludes infinite acoustic energy transfer to the duct. Nevertheless, the experimentally-measured modal pressure amplitudes $\tilde{A}_n$ do indeed become large at/near the cutoff frequency!

The method of (an infinite set of) acoustic images can be used to model sound sources inside of (perfectly reflecting – even for partially reflecting) ducts – the planar walls of the duct act like mirrors, thus virtual “images” of the sound source in the duct are produced outside of the duct, as shown in the figure below:
The pressure/particle velocity fields in proximity to the actual sound source inside the duct are determined largely by the image source(s) nearest to the actual sound source; the solution converges rapidly as the number of image sources is increased. However, accuracy in calculating the pressure / particle velocity fields far from the actual sound source (i.e. further down the duct, \( \Delta x \gg a \)) requires increasingly larger numbers of image sources to be included.

The image source technique is widely used e.g. in computational modeling of room acoustics.

The complex frequency-domain 2-D vector specific impedance/admittance, acoustic energy flow velocity, acoustic intensity and purely real, scalar energy densities, etc. inside the duct can all be computed (most easily accomplished e.g. via numerical computation…) from their definitions, for a given sound source & frequency:

\[
\tilde{z}_a(x, y, \omega) = \frac{\tilde{p}(x, y, \omega)}{\tilde{u}(x, y, \omega)} = \frac{\tilde{p}(x, y, \omega) \cdot \tilde{u}^*(x, y, \omega)}{\mid \tilde{u}(x, y, \omega) \mid^2} \quad (\Omega_a) \quad \text{and:} \quad \tilde{y}(x, y, \omega) = \frac{\tilde{u}(x, y, \omega)}{p(x, y, \omega)} \quad (\Omega_a^{-1})
\]

\[
\tilde{c}_a(x, y, \omega) = \frac{1}{\rho_o} \frac{\tilde{z}_a(x, y, \omega)}{m/s} \quad \text{and:} \quad \tilde{I}_a(x, y, \omega) = \frac{1}{2} \tilde{p}(x, y, \omega) \cdot \tilde{u}^*(x, y, \omega) \quad \text{(Watts/m}^2)\]

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\[ w_{a}^{\text{pot}}(x, y, \omega) \equiv \frac{1}{4} \frac{\sqrt{p(x, y, \omega)}}{\rho_0 c^2} \quad (\text{Joules/m}^3) \]

\[ w_{a}^{\text{kin}}(x, y, \omega) \equiv \frac{1}{4} \rho_0 \sqrt{\ddot{u}(x, y, \omega)} \quad (\text{Joules/m}^3) \]

\[ w_{a}^{\text{tot}}(x, y, \omega) \equiv w_{a}^{\text{pot}}(x, y, \omega) + w_{a}^{\text{kin}}(x, y, \omega) \quad (\text{Joules/m}^3) \]
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