Euler’s Equation for Inviscid Fluid Flow

Euler’s equation for *inviscid* (i.e. *dissipationless*) fluid flow is a special/limiting case of the more general {non-linear} Navier-Stokes equation – which expresses Newton’s 2nd law of motion for {compressible} fluid flow. The N-S eq’n, in the absence of external driving forces is:

\[ \frac{D\vec{u}(\vec{r},t)}{Dt} = -\nabla\tilde{p}(\vec{r},t) + \left( \frac{1}{\gamma} \eta + \eta_{B} \right) \nabla \left( \nabla \times \tilde{u}(\vec{r},t) \right) - \eta \nabla \times \nabla \times \tilde{u}(\vec{r},t) \]

The two *dissipative* terms on the right-hand side of the Navier-Stokes equation – a non-zero gradient of the divergence of the particle velocity \( \nabla \left( \nabla \times \tilde{u}(\vec{r},t) \right) \) and the curl of the *vorticity* of the particle velocity \( \nabla \times \tilde{u}(\vec{r},t) \) are associated with the *coefficient of shear viscosity* of the fluid \( \eta \), and the *coefficient of bulk viscosity* of the fluid \( \eta_{B} \), both of which have SI units of Pascal-seconds (Pa-s).

The time derivative term on the left-hand side of the Navier-Stokes equation, \( \frac{D\vec{u}(\vec{r},t)}{Dt} \) is the complex particle *acceleration* associated with an infinitesimal volume element \( V \) of fluid \{e.g. air\} centered on the space-time point \( (\vec{r},t) \). From dimensional analysis, note that

\[ \frac{\rho(\vec{r},t)}{Dt} \frac{D\vec{u}(\vec{r},t)}{m^3} = \frac{N}{m^3} \]  

is a force *density*. The term \( \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \tilde{u}(\vec{r},t) \cdot \nabla \) is known as the *convective* (or *substantive*, aka *material*) derivative, computed from a *stationary* observer’s reference frame, e.g. fixed in the *laboratory*:

\[ \frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \tilde{u}_{x}(\vec{r},t) \frac{\partial}{\partial x} + \tilde{u}_{y}(\vec{r},t) \frac{\partial}{\partial y} + \tilde{u}_{z}(\vec{r},t) \frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \left( \tilde{u}(\vec{r},t) \cdot \nabla \right) \]

Euler’s equation for inviscid fluid flow is a first-order, linear, homogeneous differential equation, arising from consideration of momentum conservation in a *lossless/dissipationless* compressible fluid (liquid or gas), that in the absence of external driving forces describes the relationship between complex pressure \( \tilde{p}(\vec{r},t) \) and complex particle velocity \( \tilde{u}(\vec{r},t) \) in the compressible fluid, of volume mass density \( \rho(\vec{r},t) \left( \text{kg/m}^3 \right) \). Euler’s equation for inviscid fluid flow is thus valid for fluids where the viscosity of the fluid and/or the *conduction* of *heat* in the fluid are *both* zero \{or can both be *approximated* as being *negligible*\):

\[ \frac{\rho(\vec{r},t)}{Dt} \frac{D\tilde{u}(\vec{r},t)}{m^3} = \rho(\vec{r},t) \left( \frac{\partial\tilde{u}(\vec{r},t)}{\partial t} + (\tilde{u}(\vec{r},t) \cdot \nabla)\tilde{u}(\vec{r},t) \right) = -\nabla\tilde{p}(\vec{r},t) \]
Inviscid fluid flow in a compressible liquid or gas occurs whenever the magnitude of inertial forces $\vec{F}_{\text{inertial}}(\vec{r},t)$ acting on an infinitesimal volume element $V$ of the fluid centered on the point $\vec{r}$ in the fluid are large in comparison to the dissipative forces $\vec{F}_{\text{viscous}}(\vec{r},t)$ acting on that fluid, e.g. a fluid with high Reynolds number: $R_e = \frac{|\vec{F}_{\text{inertial}}(\vec{r},t)|}{|\vec{F}_{\text{viscous}}(\vec{r},t)|} \gg 1$. “Free” air, well away from any bounding/confining surfaces is one such example of an inviscid fluid.

In analogy with electric charge conservation, the mass continuity equation for fluid flow describes conservation of mass at every space-time point $(\vec{r},t)$ within the volume $V$ of the fluid:

$$\frac{\partial \rho(\vec{r},t)}{\partial t} + \nabla \cdot (\rho(\vec{r},t) \vec{u}(\vec{r},t)) = 0$$

or:

$$\frac{\partial \rho(\vec{r},t)}{\partial t} + \nabla \cdot \vec{J}_a(\vec{r},t) = 0$$

where: $\vec{J}_a(\vec{r},t) \equiv \rho(\vec{r},t) \vec{u}(\vec{r},t)$ $(kg/m^2 \cdot s)$ is the 3-D vector acoustic mass current density.

For “everyday” complex sound fields $\tilde{S}(\vec{r},t)$ in air (at NTP) that we are considering in this course (in the audio frequency range: $20 \text{ Hz} \leq f \leq 20 \text{ KHz}$), typical sound pressure levels are:

$$\text{SPL}(\vec{r},t) = L_p(\vec{r},t) = 20 \log_{10} \left( \frac{|\tilde{p}(\vec{r},t)|}{p_o} \right) \leq 134 \text{ dB}.$$

The reference sound over-pressure amplitude is $p_o \equiv 2 \times 10^{-5} \text{ RMS Pascals} \left( = \text{ RMS N/m}^2 \right)$ in {bone-dry} air at NTP, and we have shown in a previous P406POM lecture note that a sound over-pressure amplitude of $|\tilde{p}| = 1.0 \text{ RMS Pascals}$ corresponds to a sound pressure level of $\text{SPL} = L_p = 20 \log_{10} \left( \frac{|\tilde{p}|}{p_o} \right) = 94 \text{ dB} \ll 134 \text{ dB}$ in {bone-dry} air at NTP. Note that a sound over-pressure amplitude of $|\tilde{p}| = 1.0 \text{ RMS Pascals}$ is $\ll$ than the ambient atmospheric pressure $P_{\text{atm}} = 1.013 \times 10^5 \text{ Pascals}$ at NTP, or: $|\tilde{p}|/P_{\text{atm}} = 10^{-5}$. A sound over-pressure amplitude that is as large as the atmospheric pressure itself, $|\tilde{p}(\vec{r},t)| = P_{\text{atm}} = 1.013 \times 10^5 \text{ RMS Pascals}$ corresponds to an almost unimaginable sound pressure level of $\text{SPL} = L_p = 20 \log_{10} \left( \frac{P_{\text{atm}}}{p_o} \right) = 194 \text{ dB}$! (Note that an over-pressure amplitude of $|\tilde{p}_{\text{pain}}(\vec{r},t)| = 20 \text{ RMS Pascals}$ corresponds to a sound pressure level of $\text{SPL} = L_p = 20 \log_{10} \left( \frac{|\tilde{p}_{\text{pain}}|}{p_o} \right) = 120 \text{ dB}$, which is the threshold for pain... )

Non-linear effects in air become increasingly noticeable at over-pressure amplitudes greater than $|\tilde{p}_{\text{nl}}(\vec{r},t)| = 100 \text{ RMS Pascals} \ll P_{\text{atm}} = 1.013 \times 10^5 \text{ Pascals}$, which corresponds to a sound pressure level of $\text{SPL} = L_p = 20 \log_{10} \left( \frac{|\tilde{p}_{\text{nl}}|}{p_o} \right) = 134 \text{ dB}$ (See graph below).
The \textit{non-linear} response in air for \textbf{large} pressure variations (SPL’s $\geq 134$ dB) arises from the \textit{non-linear} relation between the pressure and the density of air. For \textit{adiabatic} changes in air pressure (relevant for sound propagation in air for audio frequency sounds \textit{i.e.} $f < 20$ KHz): $P(\vec{r},t) = P_{\text{atm}} + p(\vec{r},t) = \text{constant} \times \rho^\gamma(\vec{r},t)$ \{where for air, $\gamma \equiv C_p/C_v = 7/5 = 1.4$\}. The relation between \{absolute\} pressure $P(\vec{r},t)$ and volume mass density $\rho^\gamma(\vec{r},t)$ of air is shown in the figure below, where equilibrium \textit{(i.e. no sound is present)} $P_{\text{atm}} = P_o$ and $\rho_{\text{atm}} = \rho_o$:

We can express the instantaneous absolute pressure $P(\vec{r},t)$ as a Taylor series expansion about the equilibrium pressure $P_{\text{atm}} = P_o$ and mass density $\rho_{\text{atm}} = \rho_o$ configuration:

$$P(\vec{r},t) = P_o + \frac{\partial P(\vec{r},t)}{\partial \rho(\vec{r},t)}|_{\rho = \rho_o} \left( \rho(\vec{r},t) - \rho_o \right) + \frac{1}{2} \frac{\partial^2 P(\vec{r},t)}{\partial \rho^2(\vec{r},t)}|_{\rho = \rho_o} \left( \rho(\vec{r},t) - \rho_o \right)^2 + ...$$

For \textbf{small} pressure variations $\left( |\delta p(\vec{r},t)| \ll P_{\text{atm}} \right)$ to \textbf{first} order, a \textbf{linear} relationship exists between over-pressure $p(\vec{r},t)$ and the volume mass density $\rho(\vec{r},t)$ for air:

$$p(\vec{r},t) = P(\vec{r},t) - P_o = \frac{\partial P(\vec{r},t)}{\partial \rho(\vec{r},t)}|_{\rho = \rho_o} \delta \rho(\vec{r},t)$$
A mathematical statement associated with the conservation of mass within an infinitesimal volume element $V$ of air of equilibrium volume $V_0$ is given by:  
\[ \rho V = \rho_0 V_0 = \text{constant}. \]

Thus, the \textit{volumetric strain} (relevant for sound propagation in air) is:  
\[ \delta V / V = -\delta \rho / \rho \]
or:  
\[ \delta \rho \bigg|_{\rho = \rho_0} = -\rho_0 \left( \delta V / V \right), \]

hence to \textit{first} order the over-pressure:

\[ p = \delta P = P - P_o = \frac{\partial P}{\partial \rho} \bigg|_{\rho = \rho_o} \delta \rho = -\rho_o \frac{\partial P}{\partial \rho} \bigg|_{\rho = \rho_o} \frac{\delta V}{V} = -B \frac{\delta V}{V} \]

where $B = \rho_o \frac{\partial P}{\partial \rho} \bigg|_{\rho = \rho_o}$ is the \textit{adiabatic bulk modulus} of air \{e.g. @ NTP\}.

However, for \textit{adiabatic} changes, the absolute air pressure $P = \text{constant} \times \rho^\gamma$ and thus:

\[ B = \rho_o \frac{\partial P}{\partial \rho} \bigg|_{\rho = \rho_o} = \gamma P_o, \]

hence:

\[ p = \delta P = \frac{\partial P}{\partial \rho} \bigg|_{\rho = \rho_o} \delta \rho = -\rho_o \frac{\partial P}{\partial \rho} \bigg|_{\rho = \rho_o} \frac{\delta V}{V} = -B \frac{\delta V}{V} = +B \frac{\delta \rho}{\rho} = \gamma P_o \left( \frac{\rho - \rho_o}{\rho_o} \right) = \gamma P_o \cdot s \]

The fractional change in volume mass density is known as the \textit{condensation}:  
\[ s = \frac{\delta \rho}{\rho} = \frac{(\rho - \rho_o)}{\rho_o} \]

Thus, for “everyday” audio sound over-pressure amplitudes $|\tilde{p}(\vec{r},t)| \ll 100 \text{ RMS Pascals}$ \{SPL \ll 134 dB\}, the response of air as a medium for sound propagation is very nearly \textit{linear}.

This in turn implies that for “everyday” sound over-pressure amplitudes, the volume mass density of air at NTP is nearly \textit{constant}, \textit{i.e.} $|\tilde{\rho}(\vec{r},t)| = \rho_o = 1.204 \text{ kg/m}^3$ \{\textit{i.e.} $|\tilde{s}(\vec{r},t)| = 0$\}.

However, for “everyday” audio sound over-pressure amplitudes, with \textit{small} pressure variations $|\tilde{p}(\vec{r},t)| \ll P_o$, since: $\tilde{\rho}(\vec{r},t) = \rho_o + \tilde{\rho}_o(\vec{r},t)$, thus:  
\[ \tilde{\rho}_o(\vec{r},t) = \delta \tilde{\rho}(\vec{r},t) = \tilde{\rho}(\vec{r},t) - \rho_o \]

$\left( \rho_o(\vec{r},t) \ll \rho_o \right)$ is the \{incremental\} volume mass density “amplitude” associated with the presence of the acoustic sound field, the time derivatives $\partial \tilde{\rho}(\vec{r},t)/\partial t = \partial \tilde{\rho}_o(\vec{r},t)/\partial t \neq 0$ and $\partial \tilde{s}(\vec{r},t)/\partial t \neq 0$.

However, for $|\tilde{\rho}_o(\vec{r},t)| \ll \rho_o$, the \textit{non-linear} $\tilde{\nabla} \cdot (\tilde{\rho}(\vec{r},t)\tilde{u}(\vec{r},t))$ term in the \textit{mass continuity equation} can be \textit{linearized}:

\[ \tilde{\nabla} \cdot (\tilde{\rho}(\vec{r},t)\tilde{u}(\vec{r},t)) = \tilde{\nabla} \cdot \{ \rho_o + \tilde{\rho}_o(\vec{r},t) \} \tilde{u}(\vec{r},t) \]

\[ = \rho_o \tilde{\nabla} \cdot \tilde{u}(\vec{r},t) + \tilde{\nabla} \cdot \tilde{\rho}_o(\vec{r},t) \tilde{u}(\vec{r},t) \]

\[ = \rho_o \tilde{\nabla} \cdot \tilde{u}(\vec{r},t) \]

\[ \text{neglect} \]
Hence, for “everyday” audio sound fields, the **linearized mass continuity equation** is:

\[
\frac{\partial \tilde{\rho}(\vec{r}, t)}{\partial t} + \rho_o \nabla \cdot \vec{u}(\vec{r}, t) = 0
\]

Note also that for “everyday” audio sound fields, the **linearized** complex acoustic mass current density is:

\[
\tilde{J}_a(\vec{r}, t) = \rho_o \tilde{\rho}(\vec{r}, t) (kg/m^2-s).
\]

Likewise, for “everyday” audio sound fields, the **non-linear** Euler equation can likewise be **linearized**. For \( |\tilde{\rho}(\vec{r}, t)| \ll \rho_o \), with \( \tilde{\rho}(\vec{r}, t) = \rho_o + \tilde{\rho}_a(\vec{r}, t) \) we first make the approximation:

\[
\rho_o \frac{D\tilde{u}(\vec{r}, t)}{Dt} = \rho_o \left( \frac{\partial \tilde{u}(\vec{r}, t)}{\partial t} + \left( \tilde{u}(\vec{r}, t) \cdot \nabla \right) \tilde{u}(\vec{r}, t) \right)
\]

A second approximation that we now make for “everyday” audio sound fields is that it can be shown that the magnitude of the **non-linear** term \( \left( \tilde{u}(\vec{r}, t) \cdot \nabla \right) \tilde{u}(\vec{r}, t) \) is very small in comparison to the magnitude of the \( \partial \tilde{u}(\vec{r}, t)/\partial t \) term, and hence can be neglected. Thus, the **linearized** version of Euler’s equation, valid for \( SPL \ll 134 \text{ dB} \) (over-pressure amplitudes \( |\tilde{p}(\vec{r}, t)| \ll 100 \text{ RMS Pascals} \)) becomes:

\[
\rho_o \frac{\partial \tilde{u}(\vec{r}, t)}{\partial t} = -\nabla \tilde{p}(\vec{r}, t)
\]

or:

\[
\frac{\partial \tilde{u}(\vec{r}, t)}{\partial t} = -\frac{1}{\rho_o} \nabla \tilde{p}(\vec{r}, t)
\]

The **Helmholtz Theorem** tells us that the vectorial nature of an arbitrary vector field \( \vec{F}(\vec{r}) \) is fully-specified/unique if a.) \( \lim_{r \to \infty} \vec{F}(\vec{r}) \to 0 \) and b.) the **divergence** and the **curl** of \( \vec{F}(\vec{r}) \) are both known, i.e. \( \nabla \cdot \vec{F}(\vec{r}) = C(\vec{r}) \) and \( \nabla \times \vec{F}(\vec{r}) = D(\vec{r}) \), with the restriction that \( \nabla \cdot \left( \nabla \times \vec{F}(\vec{r}) \right) = 0 \), since the divergence of the curl of any vector field is always zero.

For the 3-D particle velocity \( \tilde{u}(\vec{r}, t) \) associated with sound waves propagating in an inviscid fluid such as air, for “everyday” over-pressure amplitudes of \( |\tilde{p}(\vec{r}, t)| \ll 100 \text{ RMS Pascals} \), we showed above that the **linearized** mass continuity equation (expressing conservation of mass), tells us that the spatial divergence of the 3-D particle velocity field is equal to the negative of the normalized (aka fractional) time rate of change of the volume mass density:

\[
\nabla \cdot \tilde{u}(\vec{r}, t) = -\frac{1}{\rho_o} \frac{\partial \tilde{\rho}(\vec{r}, t)}{\partial t}
\]
What is the **curl** of the 3-D particle velocity field, \( \vec{\nabla} \times \vec{u}(\vec{r}, t) = ?? \)? Physically, the **curl** of a **velocity** field is often associated *e.g.* with a **rotation** and/or a velocity **shear** — such as the velocity field \( \vec{v}(\vec{r}, t) \) associated with a whirlpool, or a vortex in water. For this reason, the **curl** of a velocity field \( \nabla \times \vec{v}(\vec{r}, t) \) is sometimes known as/called the **vorticity**.

However, in an **inviscid** fluid (*i.e.* one which is **dissipationless**/has **zero** viscosity) such as air, the vorticity \( \nabla \times \vec{v}(\vec{r}, t) \equiv 0 \), because an **inviscid** fluid cannot support velocity **shears** and/or **vortices** in the **inviscid** fluid. We can explicitly show/prove that \( \nabla \times \vec{u}(\vec{r}, t) = 0 \) for “everyday” audio sound over-pressure amplitudes in air at NTP of \( |\tilde{p}(\vec{r}, t)| \ll 100 \text{ RMS Pascals} \). First, we take the partial derivative of \( \nabla \times \vec{u}(\vec{r}, t) \) with respect to time:

\[
\frac{\partial}{\partial t} \left( \nabla \times \vec{u}(\vec{r}, t) \right) = \nabla \times \frac{\partial \vec{u}(\vec{r}, t)}{\partial t}
\]

However, the Euler equation for inviscid fluid flow is:

\[
\frac{\partial}{\partial t} \left( \nabla \times \vec{u}(\vec{r}, t) \right) = \nabla \times \frac{\partial \vec{u}(\vec{r}, t)}{\partial t} = -\frac{1}{\rho_o} \nabla \times \tilde{\nabla} \tilde{p}(\vec{r}, t)
\]

However, the **curl** of the **gradient** of any **arbitrary** scalar field \( f(\vec{r}, t) \) is also **always** zero, *i.e.* \( \nabla \times \nabla f(\vec{r}, t) \equiv 0 \), thus:

\[
\frac{\partial}{\partial t} \left( \nabla \times \vec{u}(\vec{r}, t) \right) = \nabla \times \frac{\partial \vec{u}(\vec{r}, t)}{\partial t} = -\frac{1}{\rho_o} \left( \nabla \times \nabla \tilde{p}(\vec{r}, t) \right) = 0
\]

This tells us that: \( \nabla \times \vec{u}(\vec{r}, t) = \text{constant} \neq fcn(t) \). Thus, if for any time \(-\infty < t < +\infty\), there is **no** vorticity in the inviscid fluid \( \nabla \times \vec{u}(\vec{r}, t) = 0 \), then it must remain \( 0 \) for all time. Q.E.D.

If we take the time derivative of both sides of the {linearized} mass continuity equation, and the divergence of both sides of the {linearized} Euler equation:

\[
\nabla \cdot \frac{\partial \vec{u}(\vec{r}, t)}{\partial t} = -\frac{1}{\rho_o} \nabla \cdot \frac{\partial \tilde{p}(\vec{r}, t)}{\partial t} = -\frac{1}{\rho_o c^2} \frac{\partial^2 \tilde{p}(\vec{r}, t)}{\partial t^2}
\]

and: \( \nabla \cdot \frac{\partial \vec{u}(\vec{r}, t)}{\partial t} = -\frac{1}{\rho_o} \nabla \cdot \tilde{\nabla} \tilde{p}(\vec{r}, t) = -\frac{1}{\rho_o} \nabla^2 \tilde{p}(\vec{r}, t) \)

and then using the {linearized} adiabatic relationship between complex overpressure, \( \tilde{p} \) and mass density, \( \tilde{\rho}(\vec{r}, t) = \frac{1}{\epsilon} \tilde{p}(\vec{r}, t) \), we also have the relation: \( \partial \tilde{\rho}(\vec{r}, t) / \partial t = \frac{1}{\epsilon} \partial \tilde{p}(\vec{r}, t) / \partial t \). Hence, we obtain the {linearized} wave equation for complex overpressure:

\[
\nabla^2 \tilde{p}(\vec{r}, t) - \frac{1}{c^2} \frac{\partial^2 \tilde{p}(\vec{r}, t)}{\partial t^2} = 0
\]
If we now take the spatial gradient of both sides of the linearized mass continuity equation, and the time derivative of both sides of the linearized Euler equation, and again use the \{linearized\} adiabatic relationship between complex overpressure, $\hat{p}$ and mass density, $\hat{\rho} (\vec{r}, t) = \frac{1}{\gamma} \hat{p} (\vec{r}, t)$, we also have the relation: $\nabla \hat{\rho} (\vec{r}, t) = \frac{1}{\gamma} \nabla \hat{p} (\vec{r}, t)$, then:

$$
\nabla \left( \nabla \cdot \hat{\mathbf{u}} (\vec{r}, t) \right) = - \frac{1}{\rho_o} \frac{\partial \nabla \hat{\rho} (\vec{r}, t)}{\partial t} = - \frac{1}{\rho_o} \frac{\partial \nabla \hat{p} (\vec{r}, t)}{\partial t} \quad \text{and:} \quad \frac{\partial^2 \hat{u} (\vec{r}, t)}{\partial t^2} = - \frac{1}{\rho_o} \frac{\partial \nabla \hat{p} (\vec{r}, t)}{\partial t}
$$

Combining these two equations, we obtain:

$$
\nabla \left( \nabla \cdot \hat{\mathbf{u}} (\vec{r}, t) \right) = - \frac{1}{\rho_o} \frac{\partial \nabla \hat{p} (\vec{r}, t)}{\partial t} = \frac{1}{c^2} \frac{\partial^2 \hat{u} (\vec{r}, t)}{\partial t^2}
$$

If the complex vector acoustic particle velocity field is \textit{irrotational} (i.e. $\nabla \times \hat{\mathbf{u}} (\vec{r}, t) = 0$), then using the vector relation $\nabla^2 \hat{\mathbf{u}} = \nabla \left( \nabla \cdot \hat{\mathbf{u}} \right) - \nabla \times \left( \nabla \times \hat{\mathbf{u}} \right) = \nabla \left( \nabla \cdot \hat{\mathbf{u}} \right)$, we also obtain the \{linearized\} wave equation for complex vector particle velocity:

$$
\nabla^2 \hat{\mathbf{u}} (\vec{r}, t) - \frac{1}{c^2} \frac{\partial^2 \hat{\mathbf{u}} (\vec{r}, t)}{\partial t^2} = 0
$$

\textbf{The Complex Particle Velocity Potential, $\hat{\Phi}_u (\vec{r}, t)$}

Since an inviscid (i.e. dissipationless) fluid does not support vorticity, i.e. $\nabla \times \hat{\mathbf{u}} (\vec{r}, t) = 0$ then since the \textit{curl} of the \textit{gradient} of any \textit{arbitrary} scalar field $f (\vec{r}, t)$ is also \textit{always} zero, i.e. $\nabla \times \nabla f (\vec{r}, t) \equiv 0$, we can write $\hat{\mathbf{u}} (\vec{r}, t) = \nabla \hat{\Phi}_u (\vec{r}, t)$, where $\hat{\Phi}_u (\vec{r}, t)$ is the \textbf{complex particle velocity potential} associated with $\hat{\mathbf{u}} (\vec{r}, t)$. Then $\nabla \times \nabla \hat{\Phi}_u (\vec{r}, t) = 0$.

Note that since $\hat{\mathbf{u}} (\vec{r}, t)$ and the gradient operator $\nabla \equiv \partial / \partial x \hat{x} + \partial / \partial y \hat{y} + \partial / \partial z \hat{z}$ \{in Cartesian coordinates\} have SI units of $m/s$ and $m^{-1}$ respectively, the complex velocity potential $\hat{\Phi}_u (\vec{r}, t)$ has SI units of $m^2/s$. Physically, note also that lines/contours \{and/or 3-D surfaces\} of constant $\hat{\Phi}_u (\vec{r}, t) = K = k + i\kappa = \text{constant}$ are thus \{complex!\} “\textit{equipotentials}”, which are \{everywhere\} \textit{perpendicular} to the complex particle velocity $\hat{\mathbf{u}} (\vec{r}, t)$.

Note additionally that $\hat{\Phi}_u (\vec{r}, t)$ with $\nabla \times \hat{\mathbf{u}} (\vec{r}, t) = \nabla \times \nabla \hat{\Phi}_u (\vec{r}, t) = 0$ is the acoustic analog of the electrostatic potential $\Phi_e (\vec{r})$ associated with the electrostatic field $\vec{E} (\vec{r}) \equiv -\nabla \Phi_e (\vec{r})$, since in \textit{electrostatics} $\nabla \times \vec{E} (\vec{r}) \equiv -\nabla \times \nabla \Phi_e (\vec{r}) = 0$ \{whereas in \textit{electrodynamics}, $\nabla \times \vec{E} (\vec{r}, t) \equiv -\partial \vec{B} (\vec{r}, t) / \partial t \neq 0 \}$. 

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Exploiting the analog of the concept of electrical “voltage” \(-i.e.\) a difference in electrical potential \(\Delta \Phi_{\text{ electro}}^{\text{ b-a}} \equiv \Phi_{\text{ electro}}^{\text{ b}} - \Phi_{\text{ electro}}^{\text{ a}} = \int_{a}^{b} \nabla \Phi_{\text{ electro}} \cdot d \ell = -\int_{a}^{b} \vec{E} \cdot d \ell\) we can also define a complex particle velocity potential \textit{difference} (aka particle velocity “voltage”) as:

\[
\Delta \Phi_{u}^{\text{ b-a}} (t) \equiv \Phi_{u}^{\text{ b}} (t) - \Phi_{u}^{\text{ a}} (t) = \int_{a}^{b} \nabla \Phi_{u} (\vec{r}, t) \cdot d \ell = -\int_{a}^{b} \vec{u} (\vec{r}, t) \cdot d \ell
\]

From the mass continuity equation: \(\nabla \cdot \vec{u} (\vec{r}, t) = -\frac{1}{\rho_{0}} \left( \frac{\partial \tilde{\rho} (\vec{r}, t)}{\partial t} \right)\) and: \(\vec{u} (\vec{r}, t) = \nabla \Phi_{u} (\vec{r}, t)\), then for “everyday” audio sound over-pressure amplitudes in {bone-dry} air at NTP of \(|\tilde{p} (\vec{r}, t)| \ll 100 \text{ RMS Pascals} \} \{\text{SPL} \ll 134 \text{ dB}\}, \) then: \(\nabla \cdot \nabla \Phi_{u} (\vec{r}, t) = -\frac{1}{\rho_{0}} \left( \frac{\partial \tilde{\rho} (\vec{r}, t)}{\partial t} \right)\), which can be written as \(\nabla^{2} \Phi_{u} (\vec{r}, t) = -\frac{1}{\rho_{0}} \left( \frac{\partial \tilde{\rho} (\vec{r}, t)}{\partial t} \right)\); this is Poisson’s equation for the complex particle velocity potential!

Thus, we can thus solve {certain classes of} acoustical physics problems simply by solving Poisson’s equation \(\nabla^{2} \Phi_{u} (\vec{r}, t) = -\frac{1}{\rho_{0}} \left( \frac{\partial \tilde{\rho} (\vec{r}, t)}{\partial t} \right)\) for the complex particle velocity potential \(\Phi_{u} (\vec{r}, t)\), subject to the boundary condition(s) \{and/or initial conditions at \(t = 0\)\} associated with the specific problem using techniques/methodology similar to that used for solving Poisson’s equation \(\nabla^{2} \Phi_{u} (\vec{r}, t) \neq 0\) in E&M problems!

Note that {again} using the {linearized} adiabatic relationship between complex overpressure and mass density, \(\tilde{\rho} (\vec{r}, t) = \frac{1}{\rho_{0}} \tilde{p} (\vec{r}, t)\) we also have: \(\frac{\partial \tilde{\rho} (\vec{r}, t)}{\partial t} = \frac{1}{\rho_{0}} \frac{\partial \tilde{p} (\vec{r}, t)}{\partial t}\). Hence for “everyday” audio sound fields, the above differential equation for the complex velocity potential can equivalently be written as: \(\nabla^{2} \Phi_{u} (\vec{r}, t) = -\frac{1}{\rho_{0} c^{2}} \left( \frac{\partial \tilde{p} (\vec{r}, t)}{\partial t} \right)\).

If \(\tilde{u} (\vec{r}, t) = \nabla \Phi_{u} (\vec{r}, t)\), the \{linearized\} Euler equation can be written as:

\[
\frac{\partial \nabla \Phi_{u} (\vec{r}, t)}{\partial t} = \nabla \frac{\partial \Phi_{u} (\vec{r}, t)}{\partial t} = -\frac{1}{\rho_{0}} \nabla \tilde{p} (\vec{r}, t),
\]

which implies that:
\[
\frac{\partial \Phi_{u} (\vec{r}, t)}{\partial t} = -\frac{1}{\rho_{0}} \tilde{p} (\vec{r}, t),
\]

and hence that:
\[
\frac{\partial^{2} \Phi_{u} (\vec{r}, t)}{\partial t^{2}} = -\frac{1}{\rho_{0}} \frac{\partial \tilde{p} (\vec{r}, t)}{\partial t}.
\]

From above, we also have:
\[
\frac{\partial \tilde{p} (\vec{r}, t)}{\partial t} = c^{2} \frac{\partial \tilde{p} (\vec{r}, t)}{\partial t},
\]

thus:
\[
\frac{\partial^{2} \Phi_{u} (\vec{r}, t)}{\partial t^{2}} = -\frac{1}{\rho_{0}} \frac{\partial \tilde{p} (\vec{r}, t)}{\partial t},
\]

but from the above Poisson equation: \(\nabla^{2} \Phi_{u} (\vec{r}, t) = \frac{1}{\rho_{0}} \frac{\partial \tilde{p} (\vec{r}, t)}{\partial t}\),

thus, we obtain the wave equation for the complex velocity potential:

\[
\nabla^{2} \Phi_{u} (\vec{r}, t) - c^{2} \frac{\partial^{2} \Phi_{u} (\vec{r}, t)}{\partial t^{2}} = 0
\]
Derivation of Euler’s Equation for Inviscid Fluid Flow from Newton’s Second Law of Motion:

We can derive Euler’s equation for inviscid fluid flow using Newton’s 2nd law of motion \( \mathbf{F}_{\text{net}} = m \mathbf{a} \) and at the same time gain some useful insight into the physical meaning of particle velocity, \( \mathbf{u}(\mathbf{r},t) \).

Consider an infinitesimal volume element \( V_o = 1(\mu m)^3 \) bounded by the surface \( S_o \) centered on the space-time point \( (\mathbf{r},t) \) \{ = center of mass of the infinitesimal volume element \( V_o \} \) containing \{bone-dry\} air at NTP, in thermal equilibrium with the air surrounding it, and with equilibrium volume mass density \( \rho_o = 1.204 \text{ kg/m}^3 \), as shown in the figure below:

![Bounding Surface, \( S_o \)](image)

Rather than work in the fixed laboratory reference frame, we deliberately choose to work in a reference frame that is co-moving with the infinitesimal volume element \( V_o \) of air. Note that the pressure \( p(\mathbf{r},t) \) associated with the infinitesimal volume element \( V_o \) as measured in the co-moving reference frame of the infinitesimal volume element \( V_o \) is the same pressure as measured in the fixed laboratory frame, this is because pressure \( p(\mathbf{r},t) \) is intrinsically a scalar quantity.

The air \{at NPT\} contained within the infinitesimal volume element \( V_o \) is at a static / equilibrium absolute pressure of one atmosphere, \( i.e. p_{\text{amb}} = 1.013 \times 10^5 \text{ Pascals} \) and a finite temperature \( T = 20^\circ C (= 293.15 \text{ K}) \). At the microscopic level, the air molecules within the infinitesimal volume element \( V_o \) each have mean thermal energy \( \langle E_{\text{mol}}^{th} \rangle = \frac{1}{2} k_B T \) where \( k_B = 1.381 \times 10^{-23} \text{ Joules/Kelvin} \) and collide randomly with each other, undergoing Brownian random-walk type motions.

Suppose that a sound wave with over-pressure amplitude \( |p(\mathbf{r},t)| \ll 100 \text{ RMS Pascals} \{ SPL \ll 134 \text{ dB} \} \) is incident on the \{initially static\} air contained within the infinitesimal volume element \( V_o \). When the over-pressure amplitude \( p(\mathbf{r},t) \) is instantaneously greater (less) than the ambient pressure \( p_{\text{amb}} \), the air contained within \( V_o \) momentarily compresses (expands), respectively. Note that conceptually, the surface \( S_o \) that bounds the infinitesimal volume element \( V_o \) is endowed with “magical” properties, in that it is a fictitious, Gaussian-type surface (e.g. as commonly used in E&M problems), the nature of the bounding surface \( S_o \) also is one which expands and/or contracts as the air contained within the infinitesimal volume element \( V_o \) expands or contracts. Operationally this means we need only keep track of linear/leading-order terms in various expansions…
Furthermore, if the nature of incident sound wave is such as to cause the air molecules within the infinitesimal volume element \( V_o \) to collectively move in a given direction, \( i.e. \) to be displaced by a collective 3-D distance \( \vec{\xi}(\vec{r},t) \) from its equilibrium position, with collective velocity \( \vec{u}(\vec{r},t) \) and collective acceleration \( \vec{a}(\vec{r},t) \), the “magical” Gaussian surface \( S_o \) co-moves with the air contained within \( V_o \).

An infinitesimal volume element of size \( e.g. \) a cubic micron \( V_o = 1(\mu m)^3 \) is statistically large enough for our purposes. The air contained within this infinitesimal volume element \( V_o \) is in thermal equilibrium with itself and with the air surrounding it. Avogadro’s number \( N_A = 6.022 \times 10^{23} \) molecules/mole and recall that one mole of \{bone-dry\} air @ NTP has mean/average molar mass of \( m_{\text{air}}^{\text{mol}} = 28.97 \text{ gms/mole} \). Thus, for a volume mass density of air \( \rho_o = 1.204 \text{ kg/m}^3 \) at NTP there are \( 24.06 \text{ cm}^3/\text{mole}, \) or \(~ 25 \text{ billion} \) molecules of air per cubic micron at NTP. The average/mean velocity vector associated with the mean/average thermal energy \( \langle U_{\text{th}}(\vec{r},t) \rangle \) of this number of air molecules contained within the infinitesimal volume element \( V_o \) is \( \langle \vec{u}_{\text{mol}}(\vec{r},t) \rangle = 0 \), however the thermal energies \( \langle E_{\text{th}}^{\text{mol}} \rangle = \frac{1}{2} k_B T = \frac{1}{2} m |\vec{u}_{\text{mol}}|^2 \) associated with individual air molecules contained within \( V_o \) may be such that individual molecules within \( V_o \) leave through the bound surface \( S_o \) via exiting through one of the top, bottom or side surfaces associated with \( S_o \). However, one of the other “magical” properties endowed with the co-moving surface \( S_o \) associated with the air contained within the infinitesimal volume element \( V_o \) is that if an air molecule leaves (enters) the bounding surface \( S_o \) at a given point \( \vec{r}_{\text{mol}} \) on one side of the volume element with velocity vector \( \vec{u}_{\text{mol}}(\vec{r}_{\text{mol}},t) \), it instantaneously enters (leaves) the surface \( S_o \) again with velocity vector \( \vec{u}_{\text{mol}}(\vec{r}_{\text{mol}}^{\text{conj}},t) \), but on the other side of the volume element, at its conjugate point \( \vec{r}_{\text{mol}}^{\text{conj}} \) relative to the center point \( (\vec{r},t) \) of the infinitesimal volume element, \( V_o \). Thus the total air mass \( m_{\text{air}} \), the average/mean linear momentum \( \langle \vec{P}_{\text{air}}(\vec{r},t) \rangle \) and the average/mean thermal energy \( \langle U_{\text{th}}(\vec{r},t) \rangle \) are all conserved by this “magical” property of the fictitious Gaussian surface \( S \) bounding the infinitesimal volume element \( V_o \).

From Newton’s 2\textsuperscript{nd} law of motion, \( \vec{F}_{\text{net}} = m \vec{a} \), we can calculate the force(s) acting on the air within the infinitesimal volume element \( V \) due to an over-pressure amplitude \( p(\vec{r},t) \). The mass of air contained within the infinitesimal volume element \( V_o \) is \( m = \rho_o V_o \) (kg). Newton’s 2\textsuperscript{nd} law tells us that \( \vec{F}_{\text{net}}(\vec{r},t) = m \vec{a}(\vec{r},t) \) or that: \( \vec{a}(\vec{r},t) = \frac{\vec{F}_{\text{net}}(\vec{r},t)}{m} = \frac{\vec{F}_{\text{mol}}(\vec{r},t)}{\rho_o V_o} \). We define the \{net\} force per unit volume acting on the infinitesimal volume element as: \( \vec{f}_{\text{net}}(\vec{r},t) = \frac{\vec{F}_{\text{net}}(\vec{r},t)}{V_o} \). Thus the acceleration \( \vec{a}(\vec{r},t) = \frac{\vec{f}_{\text{net}}(\vec{r},t)}{\rho_o} \).

Next, let us (initially) consider only the \( x \)-component of the net force due to an over-pressure \( p(\vec{r},t) \) acting on the infinitesimal volume element \( V_o \) of air, as shown in a side view in the figure below:
Note that here we must be mindful of the nature of the compressive force(s) due to the over-pressure $p(\vec{r},t)$ acting on the infinitesimal volume element $V_o$ – namely, that thermal equilibrium of the air contained within the volume $V_o$, as well as all other adjacent / neighboring infinitesimal volume elements of air must be maintained at all times during this process. The restriction that $|p(\vec{r},t)| \ll 100$ RMS Pascals $\{SPL \ll 134$ dB $\}$ for harmonic/periodic over-pressure amplitudes with frequencies in the audio range of human hearing ($20$ Hz $< f < 20$ KHz) guarantees that thermal equilibrium holds during this process. From a thermodynamic perspective, this is clearly a reversible, adiabatic, and hence isentropic process.

The infinitesimal vector area elements associated with the $x_-$ (LHS) and $x_+$ (RHS) of the infinitesimal volume element $V_o$ are: $\vec{A}_- = A\hat{n}_- = -A_o \hat{x}$ $(m^2)$ and $\vec{A}_+ = A\hat{n}_+ = +A_o \hat{x}$ $(m^2)$.

Note that the unit normal vectors $\hat{n}_- = -\hat{x}$ and $\hat{n}_+ = +\hat{x}$ associated with these two surfaces, by convention, point outward from/perpendicular to the surface $S_o$.

The $x$-force acting on the LHS surface located at $x_-$ is: $\vec{F}_- = +F_+ \hat{x} = -p_x \vec{A}_- = +p_x A_o \hat{x}$.

The $x$-force acting on the RHS surface located at $x_+$ is: $\vec{F}_+ = -F_- \hat{x} = -p_x \vec{A}_+ = -p_x A_o \hat{x}$.

The net $x$-force acting on the infinitesimal volume element $V$ is: $\vec{F}_{net} = \vec{F}_+ + \vec{F}_- = -(p_+ - p_-) A_o \hat{x}$.

The net $x$-force per unit volume acting on the infinitesimal volume element $V_o = A_o \cdot \Delta x$ is:

$$\vec{f}_{net,\vec{x}} = \frac{\vec{F}_{net}}{V_o} = \frac{-(p_+ - p_-) A_o \hat{x}}{A_o \cdot \Delta x} = -\frac{\Delta p}{\Delta x} \hat{x}$$

In the limit that the volume $V_o$ of the infinitesimal volume element $\to 0$:

$$\vec{f}_{net,\vec{x}}(\vec{r},t) = \left. -\frac{\partial p(\vec{r},t)}{\partial x} \right\} \hat{x}$$

We can repeat this analysis for the $y$- and $z$-components of the net force per unit volume due to the overpressure amplitude acting on the infinitesimal volume element $V_o$ of air, the results are similar:

$$\vec{f}_{net,\vec{y}}(\vec{r},t) = \left. -\frac{\partial p(\vec{r},t)}{\partial y} \right\} \hat{y} \quad \text{and} \quad \vec{f}_{net,\vec{z}}(\vec{r},t) = \left. -\frac{\partial p(\vec{r},t)}{\partial z} \right\} \hat{z}$$
The total net 3-D vector force per unit volume is therefore:

\[ \vec{f}_{\text{net}}(\vec{r}, t) = f_{\text{net}}^x(\vec{r}, t) \hat{x} + f_{\text{net}}^y(\vec{r}, t) \hat{y} + f_{\text{net}}^z(\vec{r}, t) \hat{z} \]

\[ = -\frac{\partial p(\vec{r}, t)}{\partial x} \hat{x} - \frac{\partial p(\vec{r}, t)}{\partial y} \hat{y} - \frac{\partial p(\vec{r}, t)}{\partial z} \hat{z} = -\left( \frac{\partial}{\partial x} \hat{x} - \frac{\partial}{\partial y} \hat{y} - \frac{\partial}{\partial z} \hat{z} \right) p(\vec{r}, t) = -\vec{v} p(\vec{r}, t) \]

Thus we have: \( \vec{a}(\vec{r}, t) = \frac{\vec{f}(\vec{r}, t)}{\rho_o} \) and: \( \vec{f}_{\text{net}}(\vec{r}, t) = \vec{f}(\vec{r}, t) = -\vec{v} p(\vec{r}, t) \), hence:

\( \vec{a}(\vec{r}, t) = -\vec{v} p(\vec{r}, t)/\rho_o \). Recall that (for \( |p(\vec{r}, t)| \ll 100 \text{ RMS Pascals} \{ \text{SPL} \ll 134 \text{ dB} \} \),

the particle acceleration \( \vec{a}(\vec{r}, t) \) is the time rate of change of the particle velocity \( \vec{u}(\vec{r}, t) \),

i.e. \( \vec{a}(\vec{r}, t) = \frac{\partial \vec{u}(\vec{r}, t)}{\partial t} \), hence we obtain Euler’s equation for inviscid fluid flow, valid for air

with \( |p(\vec{r}, t)| \ll 100 \text{ RMS Pascals} \{ \text{SPL} \ll 134 \text{ dB} \} \):

\[ \vec{a}(\vec{r}, t) = \frac{\partial \vec{u}(\vec{r}, t)}{\partial t} = -\frac{1}{\rho_o} \vec{v} p(\vec{r}, t) \quad \text{Q.E.D.} \]

“Complexifying” this equation, we have:

\[ \vec{a}(\vec{r}, t) = \frac{\partial \vec{u}(\vec{r}, t)}{\partial t} = -\frac{1}{\rho_o} \vec{v} \tilde{p}(\vec{r}, t) \]

Although this relationship between the complex particle acceleration \( \vec{a}(\vec{r}, t) \), particle velocity \( \vec{u}(\vec{r}, t) \) and complex pressure \( \tilde{p}(\vec{r}, t) \) was derived in the co-moving/center-of-mass reference frame associated with the infinitesimal volume element \( V_o \) centered on the space-time point \( (\vec{r}, t) \), superimposed on top of a static pressure field \( p_{\text{atm}} = 1.013 \times 10^5 \text{ Pascals} \), it can be seen that for small, harmonic/periodic over-pressure amplitude variations, e.g. \( \tilde{p}(\vec{r}, t) = \tilde{p}_o(\vec{r}) e^{i\omega t} \)

with \( |\tilde{p}(\vec{r}, t)| \ll p_{\text{atm}} \) that each of these quantities are the same in the laboratory reference frame.

We can now also see that the complex particle displacement \( \vec{z}(\vec{r}, t) \) (m) \{from equilibrium position\}, complex particle velocity \( \vec{u}(\vec{r}, t) = \partial \vec{z}(\vec{r}, t)/\partial t \) (m/s) and complex particle acceleration \( \vec{a}(\vec{r}, t) = \partial \vec{u}(\vec{r}, t)/\partial t \) (m/s^2) are associated with the collective, random-thermal energy-averaged-out motion of the air molecules contained within the infinitesimal volume element \( V_o \) bounded by the {co-moving} surface \( S_o \) centered on the space-time point \( (\vec{r}, t) \).
Complex Sound Fields $\tilde{S}(\vec{r}, t)$:

The acoustical physics properties associated with an arbitrary “everyday” audio complex sound field $\tilde{S}(\vec{r}, t)$ can be completely/uniquely determined at the space-time point $(\vec{r}, t)$ by measuring two physical quantities associated with the complex sound field:

(a.) the complex over-pressure $\tilde{p}(\vec{r}, t)$ at the space-time point $(\vec{r}, t)$ - a scalar quantity, and

(b.) the complex particle velocity $\tilde{u}(\vec{r}, t)$ at the space-time point $(\vec{r}, t)$ - a 3-D vector quantity with: $\lim_{\vec{r} \to \infty} \tilde{u}(\vec{r}) \to 0$, $\tilde{\nabla} \cdot \tilde{u}(\vec{r}, t) = -\frac{1}{\rho_0} \left( \frac{\partial \tilde{p}(\vec{r}, t)}{\partial t} \right)$ and: $\tilde{\nabla} \times \tilde{u}(\vec{r}, t) = 0$ \{or = constant\}.

Complex Sound Field Quantities: Working in the Time-Domain vs. the Frequency-Domain

It is extremely important whenever working with any/all complex sound field quantities to understand/distinguish as to whether one is working with such quantities in the time-domain vs. the frequency-domain – they are not the same/identical…

Complex quantities in the time-domain vs. their frequency-domain counterparts are related by Fourier transforms of each other – because time $t$ (units = seconds) and frequency $f = \omega/2\pi$ (units = 1/sec = Hz) are so-called Fourier conjugate variables of each other. We thus use the notation $\tilde{S}(\vec{r}, t)$ vs. $\tilde{S}(\vec{r}, \omega)$ to indicate a time-domain complex sound field vs. frequency-domain complex sound field at the space-point $\vec{r}$, respectively.

How do we know whether we are working in the time-domain vs. the frequency domain?

A time-dependent instantaneous voltage signal $V_{p-mic}(\vec{r}, t) = V_{o-mic}(\omega_o) \cos(\omega_o t + \varphi_p(\vec{r}, \omega_o))$, e.g. output from a pressure sensitive microphone placed at the point $\vec{r} = (x\hat{x}, y\hat{y}, z\hat{z})$ in the sound field of a loudspeaker \{located at the origin $(0,0,0)$\} and driven by a sine-wave function generator (of angular frequency $\omega_o = 2\pi f_o$) + power amplifier is a typical example of a time-domain signal – it is observable e.g. on an oscilloscope, which displays the instantaneous voltage signal $V_{p-mic}(\vec{r}, t) = V_{o-mic}(\vec{r}, \omega_o) \cos(\omega_o t + \varphi_p(\vec{r}, \omega_o))$ output from the microphone as a function of time, $t$.

We specify, for clarity/definiteness’ sake that the oscilloscope trace of the display of the $p$-mic signal $V_{p-mic}(\vec{r}, t) = V_{o-mic}(\vec{r}, \omega_o) \cos(\omega_o t + \varphi_p(\vec{r}, \omega_o))$ is triggered externally by the sync signal output from the sine-wave generator – which serves as the reference signal and thus gives physical meaning to the (overall) phase $\varphi_p(\vec{r}, \omega_o)$ of the $p$-mic signal, which is defined relative to the time-domain sine-wave voltage signal $V_{FG}(t) = V_{o-FG} \cos \omega_o t$ output from the sine-wave generator, since (by industry standard, the positive-going edge of) the TTL-level sync signal output from the sine-wave generator is used to in-phase trigger the start of the oscilloscope trace displaying the microphone signal $V_{p-mic}(\vec{r}, t) = V_{o-mic}(\vec{r}, \omega_o) \cos(\omega_o t + \varphi_p(\vec{r}, \omega_o))$. 

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Note that the instantaneous time-domain voltage signals $V_{FG}(t) = V_o^{FG} \cos \omega_o t$ and $V_{p-mic}(\vec{r},t) = V_o^{p-mic}(\vec{r},\omega_o) \cos (\omega_o t + \varphi_p(\vec{r},\omega_o))$ are purely real quantities. We can “complexify” these instantaneous time-domain quantities by adding quadrature/imaginary terms to them:

$$\tilde{V}_{FG}(t) = V_o^{FG} \cos \omega_o t + i V_o^{FG} \sin \omega_o t = V_o^{FG} (\cos \omega_o t + i \sin \omega_o t) = V_o^{FG} e^{i\omega_o t}$$

and:

$$\tilde{V}_{p-mic}(\vec{r},t) = V_o^{p-mic}(\vec{r},\omega_o) \cos (\omega_o t + \varphi_p(\vec{r},\omega_o)) + i V_o^{p-mic}(\vec{r},\omega_o) \sin (\omega_o t + \varphi_p(\vec{r},\omega_o))$$

$$= V_o^{p-mic}(\vec{r},\omega_o) \{\cos (\omega_o t + \varphi_p(\vec{r},\omega_o)) + i \sin (\omega_o t + \varphi_p(\vec{r},\omega_o))\} = V_o^{p-mic}(\vec{r},\omega_o) e^{i[\omega_o t + \varphi_p(\vec{r},\omega_o)]}$$

A {dual-channel} lock-in amplifier is manifestly a frequency-domain device that is routinely used in many types of physics experiments to simultaneously measure the real (i.e. in-phase) and imaginary/quadrature (i.e. $90^\circ$ out-of-phase) components of a complex harmonic (i.e. single-frequency) signal, relative to a reference sine-wave signal of the same angular frequency $\omega_o = 2\pi f_o$.

In the above example, we could e.g. additionally simultaneously connect the microphone’s time-domain output signal $V_{p-mic}(\vec{r},t) = V_o^{p-mic}(\vec{r},\omega_o) \cos (\omega_o t + \varphi_p(\vec{r},\omega_o))$ to the input of the lock-in amplifier and then also connect the TTL-level sync output of the sine-wave generator to the reference input of the lock-in amplifier, which is phase-locked to the actual instantaneous {time-domain} sine-wave voltage signal $V_{FG}(t) = V_o^{FG} \cos \omega_o t$ output from the sine-wave generator.

The lock-in amplifier then outputs frequency-domain real (“$X(\omega_o)$”) and imaginary (“$Y(\omega_o)$”) components of the complex p-mic signal that are respectively in-phase ($90^\circ$ out-of-phase) relative to the lock-in amplifier’s reference input signal – in this case, the TTL-level sync signal output from the sine-wave generator:

$$X(\omega_o) \equiv \text{Re}\{\tilde{V}_{p-mic}(\vec{r},\omega_o)\} = \text{Re}\{V_o^{p-mic}(\vec{r},\omega_o)e^{i\varphi_p(\vec{r},\omega_o)}\} = V_o^{p-mic}(\vec{r},\omega_o) \text{Re}\{e^{i\varphi_p(\vec{r},\omega_o)}\}$$

$$= V_o^{p-mic}(\vec{r},\omega_o) \text{Re}\{\cos \varphi_p(\vec{r},\omega_o) + i \sin \varphi_p(\vec{r},\omega_o)\} = V_o^{p-mic}(\vec{r},\omega_o) \cos \varphi_p(\vec{r},\omega_o)$$

$$Y(\omega_o) \equiv \text{Im}\{\tilde{V}_{p-mic}(\vec{r},\omega_o)\} = \text{Im}\{V_o^{p-mic}(\vec{r},\omega_o)e^{i\varphi_p(\vec{r},\omega_o)}\} = V_o^{p-mic}(\vec{r},\omega_o) \text{Im}\{e^{i\varphi_p(\vec{r},\omega_o)}\}$$

$$= V_o^{p-mic}(\vec{r},\omega_o) \text{Im}\{\cos \varphi_p(\vec{r},\omega_o) + i \sin \varphi_p(\vec{r},\omega_o)\} = V_o^{p-mic}(\vec{r},\omega_o) \sin \varphi_p(\vec{r},\omega_o)$$

Thus, we see that the lock-in amplifier outputs the real (i.e. in-phase) and imaginary (i.e. $90^\circ$ out-of-phase) components of the frequency-domain complex voltage amplitude associated with the pressure microphone’s output signal, obtained at the point $\vec{r}$ in the (complex) sound field of the loudspeaker:

$$\tilde{V}_{p-mic}(\vec{r},\omega_o) = \text{Re}\{\tilde{V}_{p-mic}(\vec{r},\omega_o)\} + i \text{Im}\{\tilde{V}_{p-mic}(\vec{r},\omega_o)\}$$

$$= V_o^{p-mic}(\vec{r},\omega_o) \cos \varphi_p(\vec{r},\omega_o) + i V_o^{p-mic}(\vec{r},\omega_o) \sin \varphi_p(\vec{r},\omega_o)$$

$$= V_o^{p-mic}(\vec{r},\omega_o) \{\cos \varphi_p(\vec{r},\omega_o) + i \sin \varphi_p(\vec{r},\omega_o)\} = V_o^{p-mic}(\vec{r},\omega_o) e^{i\varphi_p(\vec{r},\omega_o)}$$
In the 2-D Re-Im complex plane, the complex **frequency-domain** phasor diagram for complex \( \tilde{V}_{p\cdot mic}(\bar{r}, \omega_o) \) is **static** (i.e. does not rotate) and appears as shown below:

In the complex **time-domain**, the entire phasor diagram for complex \( \tilde{V}_{p\cdot mic}(\bar{r}, t) \) rotates CCW in the complex plane at angular frequency \( \omega_o \).

Please see/read Physics 406 Lect. Notes 13 Part 2 for additional details on how lock-in amplifiers work, and their use(s) in the laboratory…

Graphically, the real and imaginary **frequency-domain** components of the complex voltage amplitude signal output from the \( p\)-mic might look something like that shown in the figures below, for a **pure** (i.e. single-frequency) sine-wave signal output from the sine-wave generator + power amplifier driving a loudspeaker:

\[
\tilde{V}_{p\cdot mic}(\bar{r}, \omega_o) = V_{o\cdot p\cdot mic}(\bar{r}, \omega_o) e^{i\varphi_p(\bar{r}, \omega_o)} = V_{o\cdot p\cdot mic}(\bar{r}, \omega_o) \left\{ \cos \varphi_p(\bar{r}, \omega_o) + i \sin \varphi_p(\bar{r}, \omega_o) \right\}
\]

\[X(\omega_o) \equiv \text{Re}\{\tilde{V}_{p\cdot mic}(\bar{r}, \omega_o)\} = V_{o\cdot p\cdot mic}(\bar{r}, \omega_o) \cos \varphi_p(\bar{r}, \omega_o)\]

\[Y(\omega_o) \equiv \text{Im}\{\tilde{V}_{p\cdot mic}(\bar{r}, \omega_o)\} = V_{o\cdot p\cdot mic}(\bar{r}, \omega_o) \sin \varphi_p(\bar{r}, \omega_o)\]
Note that the angular frequency “spikes” in the above figure at \( \omega' = \omega \) associated with the real and imaginary components of the complex frequency-domain amplitude \( \tilde{V}_{p-mic} (\tilde{r}, \omega_o) \) are in fact 1-D delta-functions (in angular-frequency space), which can be mathematically represented as \( V_{p-mic} (\tilde{r}, \omega_o) \cos \phi_p (\tilde{r}, \omega_o) \cdot \delta (\omega_o - \omega) \) and \( V_{p-mic} (\tilde{r}, \omega_o) \sin \phi_p (\tilde{r}, \omega_o) \cdot \delta (\omega_o - \omega) \), respectively. Note one of the many interesting/intriguing properties of the 1-D delta function: Since \( 2 \pi f = f' \), hence \( \int_{-\infty}^{\infty} \delta (\omega_o - \omega) d\omega = \int_{-\infty}^{\infty} \delta (2\pi f_o - 2\pi f) \cdot 2\pi df = \int_{-\infty}^{\infty} \delta (2\pi f_o - f) \cdot 2\pi df \)

Note further that the 1-D delta function \( \delta (\omega_o - \omega) \) has physical units of inverse angular frequency, \( 1/\omega \text{ (i.e. sec/radian)} \) and that the 1-D delta function \( \delta (f_o - f) \) has physical units of inverse frequency, \( 1/f \text{ (i.e. seconds)} \), since the 1-D integrals \( \int_{-\infty}^{\infty} \delta (\omega_o - \omega) d\omega = 1 \) and \( \int_{-\infty}^{\infty} \delta (f_o - f) df = 1 \) are both dimensionless.

The above complex frequency-domain result(s) should be compared with their complex time-domain counterparts:

\[
\tilde{V}_{p-mic} (\tilde{r}, t) = V_{p-mic} (\tilde{r}, \omega_o) e^{i(\omega_o t + \phi_p (\tilde{r}, \omega_o))} = V_{p-mic} (\tilde{r}, \omega_o) e^{i\phi_p (\tilde{r}, \omega_o)} \cdot e^{i\omega_o t}
\]

\[
\tilde{V}_{p-mic} (\tilde{r}, t) = V_{p-mic} (\tilde{r}, \omega_o) \left\{ \cos (\omega_o t + \phi_p (\tilde{r}, \omega_o)) + i \sin (\omega_o t + \phi_p (\tilde{r}, \omega_o)) \right\}
\]

\[
X(t) \equiv \text{Re} \left\{ \tilde{V}_{p-mic} (\tilde{r}, t) \right\} = \text{Re} \left\{ V_{p-mic} (\tilde{r}, \omega_o) e^{i(\omega_o t + \phi_p (\tilde{r}, \omega_o))} \right\} = V_{p-mic} (\tilde{r}, \omega_o) \text{Re} \left\{ e^{i(\omega_o t + \phi_p (\tilde{r}, \omega_o))} \right\}
\]

\[
X(t) \equiv \text{Re} \left\{ V_{p-mic} (\tilde{r}, \omega_o) \left\{ \cos (\omega_o t + \phi_p (\tilde{r}, \omega_o)) + i \sin (\omega_o t + \phi_p (\tilde{r}, \omega_o)) \right\} \right\}
\]

\[
X(t) \equiv V_{p-mic} (\tilde{r}, \omega_o) \cos (\omega_o t + \phi_p (\tilde{r}, \omega_o))
\]

\[
Y(t) \equiv \text{Im} \left\{ \tilde{V}_{p-mic} (\tilde{r}, t) \right\} = \text{Im} \left\{ V_{p-mic} (\tilde{r}, \omega_o) e^{i(\omega_o t + \phi_p (\tilde{r}, \omega_o))} \right\} = V_{p-mic} (\tilde{r}, \omega_o) \text{Im} \left\{ e^{i(\omega_o t + \phi_p (\tilde{r}, \omega_o))} \right\}
\]

\[
Y(t) \equiv \text{Im} \left\{ V_{p-mic} (\tilde{r}, \omega_o) \left\{ \cos (\omega_o t + \phi_p (\tilde{r}, \omega_o)) + i \sin (\omega_o t + \phi_p (\tilde{r}, \omega_o)) \right\} \right\}
\]

\[
Y(t) \equiv V_{p-mic} (\tilde{r}, \omega_o) \sin (\omega_o t + \phi_p (\tilde{r}, \omega_o))
\]

As mentioned above, the frequency-domain counterparts of complex time-domain quantities such as \( \tilde{V}_{FG} (t) = V_{FG} e^{i\omega_o t} \) and \( \tilde{V}_{p-mic} (\tilde{r}, t) = V_{p-mic} (\tilde{r}, \omega_o) e^{i(\omega_o t + \phi_p (\tilde{r}, \omega_o))} \) are obtained by taking the Fourier transform of the time-domain quantities, and vice-versa.
What is a **Fourier transform**?

For _continuous_ complex time-domain functions \( \tilde{f}(t) \), the **Fourier transform** of the complex time-domain function \( \tilde{f}(t) \) to the complex frequency-domain is:

\[
\tilde{f}(\omega) \equiv \int_{-\infty}^{\infty} \tilde{f}(t)e^{-i\omega t} \, dt
\]

where \( t \) is treated as a {dummy} variable in the integration over {all} time, from \(-\infty \leq t \leq +\infty\).

The **inverse Fourier transform** of a continuous complex frequency-domain function \( \tilde{f}(\omega) \) to the time-domain is:

\[
\tilde{f}(t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega t} \, d\omega
\]

where \( \omega = 2\pi f \) is treated as a {dummy} variable in the integration over {all negative .and. positive} angular frequencies: \(-\infty \leq \omega \leq +\infty\).

Note also that the factor of \( 1/2\pi \) appears here pre-multiplying the latter integral over the angular frequency variable \( \omega \) because we are using the angular frequency \( \omega = 2\pi f \) in the integral rather than the frequency \( f \) itself as a {dummy} variable of integration – technically speaking, frequency \( f \) (sec\(^{-1}\)) and time \( t \) (seconds) are **true Fourier conjugate variables** of each other, and **not** angular frequency \( \omega = 2\pi f \) (radians/sec\(^{-1}\)) and time \( t \) (seconds).

For monochromatic/single-frequency (aka harmonic) sound fields the relationship between “generic” complex time-domain vs. complex frequency-domain quantities is simply given by

\[
\tilde{p}(t) = \tilde{p}(\omega) \cdot e^{i\omega t} = \tilde{p}(\omega) \cdot e^{i\omega_{p}(\omega)} \cdot e^{i\omega t}
\]

and:

\[
\tilde{u}(t) = \tilde{u}(\omega) \cdot e^{i\omega t} = (\tilde{u}_{x}(\omega) \hat{x} + \tilde{u}_{y}(\omega) \hat{y} + \tilde{u}_{z}(\omega) \hat{z}) \cdot e^{i\omega t} = (|\tilde{u}_{x}(\omega)| \cdot e^{i\phi_{x}(\omega)} \hat{x} + |\tilde{u}_{y}(\omega)| \cdot e^{i\phi_{y}(\omega)} \hat{y} + |\tilde{u}_{z}(\omega)| \cdot e^{i\phi_{z}(\omega)} \hat{z}) \cdot e^{i\omega t}
\]

There are several useful relations associated with Fourier transforms which we list here:

<table>
<thead>
<tr>
<th>Time-Domain:</th>
<th>Frequency Domain:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity: ( \tilde{h}(t) = a\tilde{f}(t) + b\tilde{g}(t) ) ( \Rightarrow ) ( \tilde{h}(\omega) = a\tilde{f}(\omega) + b\tilde{g}(\omega) )</td>
<td></td>
</tr>
<tr>
<td>Translation: ( \tilde{h}(t) = \tilde{f}(t - t_{0}) ) ( \Rightarrow ) ( \tilde{h}(\omega) = \tilde{f}(\omega)e^{i\omega t_{0}} )</td>
<td></td>
</tr>
<tr>
<td>Modulation: ( \tilde{h}(t) = \tilde{f}(t)e^{i\omega_{0}t} ) ( \Rightarrow ) ( \tilde{h}(\omega) = \tilde{f}(\omega - \omega_{0}) )</td>
<td></td>
</tr>
<tr>
<td>Scaling: ( \tilde{h}(t) = \tilde{f}(at) ) ( \Rightarrow ) ( \tilde{h}(\omega) = \frac{1}{</td>
<td>a</td>
</tr>
<tr>
<td>Conjugation: ( \tilde{h}(t) = \tilde{f}^{<em>}(t) ) ( \Rightarrow ) ( \tilde{h}(\omega) = \tilde{f}^{</em>}(-\omega) )</td>
<td></td>
</tr>
</tbody>
</table>
Complex Specific Acoustic Immittances - Admittance and Impedance of a Medium:

The medium (solid, liquid or gas) in which sound waves propagate has associated with it the property of how easy (or how difficult) it is for sound waves to propagate through that medium – the so-called complex specific acoustic immittances – complex specific acoustic admittance and/or complex specific acoustic impedance (the reciprocal of complex specific acoustic admittance) give us such information.

For propagation of 1-D sound waves in a medium, the complex specific acoustic immittances – i.e. collectively the complex specific acoustic admittance and/or complex specific acoustic impedance are both well-defined quantities. They are defined in analogy to the complex form of Ohm’s Law \( \tilde{V} = \tilde{I} \tilde{Z} \) as used e.g. in electrical circuit theory, since complex over-pressure \( \tilde{p} \) is the analog of complex AC voltage \( \tilde{V} \), and particle velocity \( \tilde{u} \) is the analog of complex AC electric current \( \tilde{I} \) \{Note that \( \tilde{J}_a (\vec{r}, t) \equiv \rho_s \tilde{u} (\vec{r}, t) \) (kg/s-m²) is the complex acoustic mass current density\}, whereas \( \tilde{J}_e = \tilde{I}/\tilde{A}_e = n_e q_e \tilde{v}_e = \rho_e \tilde{v}_e \) (Amp/m² = Coul/s-m²) is the complex electrical current density\}. Note also that both \( \tilde{J}_a \) and \( \tilde{J}_e \) are 3-D vector quantities.

Complex Scalar Electrical Immittances:

| Complex Electrical Impedance: | \[ \tilde{Z}_e (t; \omega) \equiv \frac{\tilde{V} (t; \omega)}{\tilde{I}_e (t; \omega)} \quad \text{(Ohms = Volts/Amps)} \] |
| Complex Electrical Admittance: | \[ \tilde{Y}_e (t; \omega) \equiv \frac{\tilde{I}_e (t; \omega)}{\tilde{V} (t; \omega)} \quad \text{(Siemens = Ohms}^{-1} = \text{Amps/Volts}) \] |

If we write out these relations using complex polar notation: \( \tilde{V} (t; \omega) = |\tilde{V} (\omega)| e^{i\varphi_\omega} \cdot e^{i\omega t} \), \( \tilde{I}_e (t; \omega) = |\tilde{I}_e (\omega)| e^{i\varphi_\omega} \cdot e^{i\omega t} \), then, noting the cancellation of \( e^{i\omega t} \) time dependence factors:

\[
\tilde{Z}_e (t; \omega) \equiv \frac{\tilde{V} (t; \omega)}{\tilde{I}_e (t; \omega)} = \left| \frac{\tilde{V} (\omega)}{\tilde{I}_e (\omega)} \right| e^{i\varphi_\omega} \cdot e^{i\omega t} = \left| \frac{\tilde{V} (\omega)}{\tilde{I}_e (\omega)} \right| e^{i\varphi_\omega} = \tilde{Z}_e (\omega)
\]

\[
\tilde{Y}_e (t; \omega) \equiv \frac{\tilde{I}_e (t; \omega)}{\tilde{V} (t; \omega)} = \left| \frac{\tilde{I}_e (\omega)}{\tilde{V} (\omega)} \right| e^{i\varphi_\omega} \cdot e^{i\omega t} = \left| \frac{\tilde{I}_e (\omega)}{\tilde{V} (\omega)} \right| e^{i\varphi_\omega} = \tilde{Y}_e (\omega)
\]

Now: \( |\tilde{Z}_e (\omega)| = 1/|\tilde{Y}_e (\omega)| \) or: \( |\tilde{Y}_e (\omega)| = 1/|\tilde{Z}_e (\omega)| \), and we see that: \( \varphi_Z (\omega) = \varphi_v (\omega) - \varphi_f (\omega) = -\varphi_f (\omega) \), hence: \( \tilde{Y}_e (\omega) = |\tilde{Y}_e (\omega)| e^{i\varphi_\omega} = \left| \frac{1}{|\tilde{Z}_e (\omega)|} \right| e^{-i\varphi_\omega} = 1/|\tilde{Z}_e (\omega)| e^{i\varphi_\omega} = 1/|\tilde{Z}_e (\omega)| \). Thus:

\[
\tilde{Z}_e (t; \omega) \equiv \frac{\tilde{V} (t; \omega)}{\tilde{I}_e (t; \omega)} = \tilde{Z}_e (\omega) = \frac{1}{\tilde{Y}_e (\omega)}
\]

and:

\[
\tilde{Y}_e (t; \omega) \equiv \frac{\tilde{I}_e (t; \omega)}{\tilde{V} (t; \omega)} = \tilde{Y}_e (\omega) = \frac{1}{\tilde{Z}_e (\omega)}
\]
Complex 3-D Vector Specific Acoustic Immitances:

\[
\text{Cmplx Spec. Acoust. Impedance: } \quad \frac{\vec{z}_a(\vec{r},t)}{\vec{u}(\vec{r},t)} = \frac{1}{\vec{y}_a(\vec{r},t)} \quad \left( \begin{array}{c}
\text{Acoustic} \\
\text{Pa-s/m} \\
\text{Ohms} \\
\text{N-s/m}^3 \equiv \text{Rayl}
\end{array} \right)
\]

\[
\text{Cmplx Spec. Acoust. Admittance: } \quad \frac{\vec{y}_a(\vec{r},t)}{\vec{p}(\vec{r},t)} = \frac{1}{\vec{z}_a(\vec{r},t)} \quad \left( \begin{array}{c}
\text{Acoustic} \\
\text{m/Pa-s} \\
\text{Siemens} \\
\text{m}^3/N-s \equiv \text{Rayl}^{-1}
\end{array} \right)
\]

Note that the complex specific acoustic immitances \( \vec{z}_a(\vec{r},t) \) and \( \vec{y}_a(\vec{r},t) = 1/\vec{z}_a(\vec{r},t) \) are 3-D vector quantities.

The complex 3-D vector specific acoustic admittance \( \vec{y}_a(\vec{r},t) \equiv \vec{u}(\vec{r},t)/\vec{p}(\vec{r},t) \) clearly is a mathematically well-defined vector quantity:

\[
\vec{y}_a(\vec{r},t) = \frac{\vec{u}_x(\vec{r},t) \hat{x} + \vec{u}_y(\vec{r},t) \hat{y} + \vec{u}_z(\vec{r},t) \hat{z}}{\vec{p}(\vec{r},t)} = \frac{\vec{u}(\vec{r},t)}{\vec{p}(\vec{r},t)} = \frac{\vec{y}_a(\vec{r},t)}{\vec{p}(\vec{r},t)},
\]

where: \( \vec{y}_a(\vec{r},t) = \frac{\vec{u}_x(\vec{r},t)}{\vec{p}(\vec{r},t)} \), \( \vec{y}_a(\vec{r},t) = \frac{\vec{u}_y(\vec{r},t)}{\vec{p}(\vec{r},t)} \), \( \vec{y}_a(\vec{r},t) = \frac{\vec{u}_z(\vec{r},t)}{\vec{p}(\vec{r},t)} \).

The complex 3-D vector specific acoustic impedance \( \vec{z}_a(\vec{r},t) \equiv \vec{p}(\vec{r},t)/\vec{u}(\vec{r},t) \) may initially seem like a mathematically less well-defined vector quantity. However, on physical/common sense grounds, we know that e.g. the magnitudes of the complex 3-D vector specific acoustic immitances, \( |\vec{y}_a(\vec{r},t)| \) and \( |\vec{z}_a(\vec{r},t)| \) must both be invariant (i.e. unchanged) under simple coordinate transformations – e.g. rotations and/or translations of the local coordinate system, as well as invariant under e.g. simple rotations of the sound source under investigation.

Consider a simple, 1-D monochromatic/single-frequency sound field – such as an acoustic traveling plane wave propagating e.g. in the local \( +\hat{x} \) direction. Then \( \vec{u}_x(\vec{r},t) = u_o e^{i(o-t-k_x)} \neq 0 \), with \( \vec{p}(\vec{r},t) = p_o e^{i(o-t-k_x)} \neq 0 \), whereas \( \vec{u}_y(\vec{r},t) = \vec{u}_z(\vec{r},t) = 0 \). The components of the complex 3-D vector specific acoustic admittance are \( \vec{y}_a(\vec{r},t) = \frac{\vec{u}(\vec{r},t)}{\vec{p}(\vec{r},t)} = u_o e^{i(o-t-k_x)} / p_o e^{i(o-t-k_x)} = u_o / p_o \neq 0 \), whereas \( \vec{y}_a(\vec{r},t) = \vec{y}_a(\vec{r},t) = 0 \).
Obviously, if we carry out e.g. a simple rotation of our local 3-D coordinate system, the individual $x, y, z$ components of $\tilde{y}_a(\vec{r},t)$ will change accordingly, however the magnitude
\[
|\tilde{y}_a(\vec{r},t)| = \sqrt{\tilde{y}_a^* (\vec{r},t) \tilde{y}_a (\vec{r},t)} = \sqrt{|\tilde{y}_a(\vec{r},t)|^2 + |\tilde{y}_a(\vec{r},t)|^2 + |\tilde{y}_a(\vec{r},t)|^2} \text{ will not change.}
\]

Likewise, the individual $x, y, z$ components of $\tilde{z}_a(\vec{r},t)$ will change accordingly under a simple rotation of our local 3-D coordinate system, however the magnitude
\[
|\tilde{z}_a(\vec{r},t)| = \sqrt{\tilde{z}_a^* (\vec{r},t) \tilde{z}_a (\vec{r},t)} = \sqrt{|\tilde{z}_a(\vec{r},t)|^2 + |\tilde{z}_a(\vec{r},t)|^2 + |\tilde{z}_a(\vec{r},t)|^2} \text{ will not change.}
\]

We thus write the complex 3-D vector \textit{specific} acoustic impedance $\tilde{z}_a(\vec{r},t)$, e.g. in Cartesian coordinates as follows:
\[
\tilde{z}_a(\vec{r},t) \equiv \tilde{z}_a(\vec{r},t) = \tilde{x}_a(\vec{r},t) \hat{x} + \tilde{y}_a(\vec{r},t) \hat{y} + \tilde{z}_a(\vec{r},t) \hat{z}
\]
\[
= \tilde{p}(\vec{r},t) \tilde{u}^*(\vec{r},t) = \tilde{p}(\vec{r},t) \tilde{u}^*(\vec{r},t) + \tilde{p}(\vec{r},t) \tilde{u}^*(\vec{r},t) = \tilde{p}(\vec{r},t) \tilde{u}^*(\vec{r},t) = \tilde{p}(\vec{r},t) \tilde{u}^*(\vec{r},t)
\]
\[
= \tilde{p}(\vec{r},t) \tilde{u}^*(\vec{r},t) \hat{x} + \tilde{u}_a(\vec{r},t) \hat{y} + \tilde{u}_a(\vec{r},t) \hat{z}
\]
\[
= \tilde{p}(\vec{r},t) \tilde{u}^*(\vec{r},t) \hat{x} + \tilde{u}_a(\vec{r},t) \hat{y} + \tilde{u}_a(\vec{r},t) \hat{z}
\]
\[
|\tilde{u}(\vec{r},t)|
\]
\[
= \tilde{p}(\vec{r},t) \tilde{u}^*(\vec{r},t) \hat{x} + \tilde{u}_a(\vec{r},t) \hat{y} + \tilde{u}_a(\vec{r},t) \hat{z}
\]
\[
|\tilde{u}(\vec{r},t)|^2 + |\tilde{u}(\vec{r},t)|^2 + |\tilde{u}(\vec{r},t)|^2
\]
\[
where: \tilde{z}_a(\vec{r},t) = \tilde{p}(\vec{r},t) \tilde{u}^*(\vec{r},t), \tilde{z}_a(\vec{r},t) = \tilde{p}(\vec{r},t) \tilde{u}^*(\vec{r},t), \tilde{z}_a(\vec{r},t) = \tilde{p}(\vec{r},t) \tilde{u}^*(\vec{r},t)
\]

Hence, the technical/mathematical issue here is the rationalization of an arbitrary, “generic” complex reciprocal 3-D vector quantity:
\[
\tilde{u}^{-1} = \frac{1}{\tilde{u}} = \frac{\tilde{u}^*}{\tilde{u} \tilde{u}^*} = \frac{\tilde{u}^*}{|\tilde{u}|^2}
\]
paralleling that which is done for an arbitrary, “generic” complex reciprocal scalar quantity:
\[
\tilde{p}^{-1} = \frac{\tilde{p}^*}{\tilde{p} \tilde{p}^*} = \frac{\tilde{p}^*}{|\tilde{p}|^2}
\]

It can be seen that indeed: $|\tilde{y}_a(\vec{r},t)| = \frac{|\tilde{u}(\vec{r},t)|}{|\tilde{p}(\vec{r},t)|} = \frac{1}{|\tilde{z}_a(\vec{r},t)|}$, and also that:

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\[
|\vec{z}_a(\vec{r},t)| = \sqrt{|\vec{z}_a(\vec{r},t)\cdot \vec{z}_a(\vec{r},t)|^2 + |\vec{z}_a(\vec{r},t)|^2 + |\vec{z}_a(\vec{r},t)|^2} \\
= \sqrt{\left(\left|\vec{p}(\vec{r},t)\right|^2 + \left|\vec{u}(\vec{r},t)\right|^2 + \left|\vec{p}(\vec{r},t)\right|^2\right)^2 + \left|\vec{p}(\vec{r},t)\right|^2} \\
= \sqrt{\left(\left|\vec{p}(\vec{r},t)\right|^2 + \left|\vec{u}(\vec{r},t)\right|^2 + \left|\vec{p}(\vec{r},t)\right|^2\right)^2} + \left|\vec{p}(\vec{r},t)\right|^2} \\
= \sqrt{\frac{\left|\vec{p}(\vec{r},t)\right|^2}{\left|\vec{u}(\vec{r},t)\right|^2}} = \frac{1}{\left|\vec{y}_a(\vec{r},t)\right|}
\]

However, we also see for the individual \(x, y, z\) components of the complex 3-D vector specific acoustic immittances that:

\[
\begin{align*}
\vec{y}_{a_x}(\vec{r},t) &\equiv \frac{\hat{u}_x(\vec{r},t)}{\vec{p}(\vec{r},t)} & \neq & \frac{1}{\vec{z}_{a_x}(\vec{r},t)} \equiv \frac{\vec{u}_x(\vec{r},t)}{\vec{p}(\vec{r},t)\hat{u}_x(\vec{r},t)} \\
\vec{y}_{a_y}(\vec{r},t) &\equiv \frac{\hat{u}_y(\vec{r},t)}{\vec{p}(\vec{r},t)} & \neq & \frac{1}{\vec{z}_{a_y}(\vec{r},t)} \equiv \frac{\vec{u}_y(\vec{r},t)}{\vec{p}(\vec{r},t)\hat{u}_y(\vec{r},t)} \\
\vec{y}_{a_z}(\vec{r},t) &\equiv \frac{\hat{u}_z(\vec{r},t)}{\vec{p}(\vec{r},t)} & \neq & \frac{1}{\vec{z}_{a_z}(\vec{r},t)} \equiv \frac{\vec{u}_z(\vec{r},t)}{\vec{p}(\vec{r},t)\hat{u}_z(\vec{r},t)}
\end{align*}
\]

Additionally, the expressions for the complex 3-D vector specific acoustic immittances:

\[
\vec{z}_a(\vec{r},t) = \frac{\vec{u}(\vec{r},t)}{\vec{p}(\vec{r},t)} = \vec{y}_{a_x}(\vec{r},t)\hat{x} + \vec{y}_{a_y}(\vec{r},t)\hat{y} + \vec{y}_{a_z}(\vec{r},t)\hat{z} = \frac{\hat{u}_x(\vec{r},t)}{\vec{p}(\vec{r},t)}\hat{x} + \frac{\hat{u}_y(\vec{r},t)}{\vec{p}(\vec{r},t)}\hat{y} + \frac{\hat{u}_z(\vec{r},t)}{\vec{p}(\vec{r},t)}\hat{z}
\]

and:

\[
\vec{z}_a(\vec{r},t) = \frac{\vec{p}(\vec{r},t)}{\vec{u}(\vec{r},t)} = \vec{z}_{a_x}(\vec{r},t)\hat{x} + \vec{z}_{a_y}(\vec{r},t)\hat{y} + \vec{z}_{a_z}(\vec{r},t)\hat{z} = \frac{\vec{p}(\vec{r},t)}{\vec{u}(\vec{r},t)}\left[\hat{u}_x(\vec{r},t)\hat{x} + \hat{u}_y(\vec{r},t)\hat{y} + \hat{u}_z(\vec{r},t)\hat{z}\right]
\]

can be seen to mathematically behave properly e.g. under arbitrary rotations of the local 3-D coordinate system, as well as for rotations of 3-D sound sources, and also for complex 3-D sound fields composed of e.g. an arbitrary superposition/linear combination of three mutually-orthogonal propagating monochromatic plane traveling waves – propagating in the \(+\hat{x}, +\hat{y}\) and \(+\hat{z}\) directions, with common scalar complex pressure, \(\vec{p}_{\text{tot}}(\vec{r},t) = \vec{p}_1(\vec{r},t) + \vec{p}_2(\vec{r},t) + \vec{p}_3(\vec{r},t)\).
Note also that both the time-domain complex pressure \( \tilde{p}(\vec{r},t) \) and the time-domain complex 3-D particle velocity \( \tilde{u}(\vec{r},t) \) associated e.g. with a single frequency (aka harmonic) sound field will in general have time dependence of the form \( e^{i\omega t} \). Thus, since the 3-D specific acoustic immittances are defined as ratios of these two quantities, the \( e^{i\omega t} \) factor in the both the numerator and the denominator of the ratios \( \tilde{y}_a(\vec{r},t) = \tilde{u}(\vec{r},t)/\tilde{p}(\vec{r},t) \) and \( \tilde{z}_a(\vec{r},t) = \tilde{p}(\vec{r},t)/\tilde{u}(\vec{r},t) \) cancels for harmonic/single-frequency complex sound fields, thus we see that the complex 3-D vector specific acoustic immittances are in fact time-independent quantities… In fact, they are manifestly frequency-domain quantities!

**Complex 3-D Specific Acoustic Immittances (for Harmonic Sound Fields):**

<table>
<thead>
<tr>
<th>Complex Specific Acoustic Impedance:</th>
<th>[ \tilde{z}_a(\vec{r},\omega) \equiv \frac{\tilde{p}(\vec{r},\omega)}{\tilde{u}(\vec{r},\omega)} = \frac{1}{\tilde{y}_a(\vec{r},\omega)} \quad (\Omega_a = Rayl) ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Complex Specific Acoustic Admittance:</td>
<td>[ \tilde{y}_a(\vec{r},\omega) \equiv \frac{\tilde{u}(\vec{r},\omega)}{\tilde{p}(\vec{r},\omega)} = \frac{1}{\tilde{z}_a(\vec{r},\omega)} \quad (\Omega_a^{-1} = Rayl^{-1}) ]</td>
</tr>
</tbody>
</table>

The time-independent complex specific acoustic immittances are 3-D vector frequency-domain quantities. Their 3-D x-y-z Cartesian frequency-domain components can be explicitly written out as:

\[
\tilde{y}_a(\vec{r},\omega) = \tilde{y}_{ax}(\vec{r},\omega)\hat{x} + \tilde{y}_{ay}(\vec{r},\omega)\hat{y} + \tilde{y}_{az}(\vec{r},\omega)\hat{z} = \frac{\tilde{u}(\vec{r},\omega)}{\tilde{p}(\vec{r},\omega)} = \frac{1}{\tilde{z}_a(\vec{r},\omega)}
\]

\[
\tilde{z}_a(\vec{r},\omega) = \tilde{z}_{ax}(\vec{r},\omega)\hat{x} + \tilde{z}_{ay}(\vec{r},\omega)\hat{y} + \tilde{z}_{az}(\vec{r},\omega)\hat{z} = \frac{\tilde{p}(\vec{r},\omega)}{\tilde{u}(\vec{r},\omega)} = \frac{1}{\tilde{y}_a(\vec{r},\omega)}
\]
Next, we explain why $\tilde{z}_a(\vec{r}, \omega)$ and $\tilde{y}_a(\vec{r}, \omega) = 1/\tilde{z}_a(\vec{r}, \omega)$ are called complex specific acoustic impedance and admittance, respectively. As mentioned above, $\tilde{z}_a(\vec{r}, \omega)$ and $\tilde{y}_a(\vec{r}, \omega) = 1/\tilde{z}_a(\vec{r}, \omega)$ are immittances specifically associated with the propagation medium. And, in order to avoid confusion, there {already} exists two other acoustic immittance quantities, known as the complex 3-D vector acoustic impedance $\tilde{Z}_a(\vec{r}, \omega)$ and the complex 3-D vector acoustic admittance $\tilde{Y}_a(\vec{r}, \omega) = 1/\tilde{Z}_a(\vec{r}, \omega)$, which are associated with the acoustics of sound waves propagating inside ducts (i.e. pipes) with cross-sectional area $S_\perp$ as defined below:

**Complex 3-D Acoustic Immittances (for Harmonic Sound Fields):**

**Complex 3-D Acoustic Impedance:**

$$\tilde{Z}_a(\vec{r}, \omega) \equiv \frac{\tilde{p}(\vec{r}, \omega)}{\tilde{u}(\vec{r}, \omega)S_\perp} = \frac{1}{\tilde{Y}_a(\vec{r}, \omega)} \left( \frac{Pa-s/m^3}{N-s/m^3} = Rayl/m^2 \right)$$

**Complex 3-D Acoustic Admittance:**

$$\tilde{Y}_a(\vec{r}, \omega) \equiv \frac{\tilde{u}(\vec{r}, \omega)S_\perp}{\tilde{p}(\vec{r}, \omega)} = \frac{1}{\tilde{Z}_a(\vec{r}, \omega)} \left( \frac{m/Pa-s}{m^3/N-s} = Rayl^{-1}-m^2 \right)$$

Note that the quantity $\tilde{U}(\vec{r}, \omega) \equiv \tilde{u}(\vec{r}, \omega)S_\perp \left( m/s \cdot m^2 = m^3/s \right)$ is known as the **volume velocity**, because of its dimensions $\left( m^3/s \right)$.

Inside a duct of cross sectional area $S_\perp$, the complex 3-D vector specific acoustic immittances $\tilde{z}_a(\vec{r}, \omega)$ and $\tilde{y}_a(\vec{r}, \omega) = 1/\tilde{z}_a(\vec{r}, \omega)$ are thus related to the complex 3-D vector immittances $\tilde{Z}_a(\vec{r}, \omega)$ and $\tilde{Y}_a(\vec{r}, \omega) = 1/\tilde{Z}_a(\vec{r}, \omega)$ by the relations:

$$\tilde{z}_a(\vec{r}, \omega) = \frac{\tilde{Z}_a(\vec{r}, \omega)}{S_\perp} \quad \text{and} \quad \tilde{y}_a(\vec{r}, \omega) = \frac{\tilde{Y}_a(\vec{r}, \omega)}{S_\perp}$$

or:

$$\frac{\tilde{Z}_a(\vec{r}, \omega)}{S_\perp} = \frac{\tilde{Z}_a(\vec{r}, \omega)}{S_\perp} \quad \text{and} \quad \frac{\tilde{Y}_a(\vec{r}, \omega)}{S_\perp} = \frac{\tilde{Y}_a(\vec{r}, \omega)}{S_\perp}$$

From the above relations, since the complex 3-D vector specific acoustic immittances $\tilde{z}_a(\vec{r}, \omega)$ and $\tilde{y}_a(\vec{r}, \omega) = 1/\tilde{z}_a(\vec{r}, \omega)$ are manifestly frequency domain quantities, we see that the complex 3-D vector acoustic immittances $\tilde{Z}_a(\vec{r}, \omega)$ and $\tilde{Y}_a(\vec{r}, \omega) = 1/\tilde{Z}_a(\vec{r}, \omega)$ are also manifestly frequency domain quantities.
Physically, just as the complex scalar electrical **impedance** $\tilde{Z}_e$ is a measure of an electrical device to **impede** the **flow** of a complex scalar AC electrical current $\tilde{I} = \tilde{J}_e \cdot \tilde{S}_\perp (C/s)$ when a complex scalar AC voltage $\tilde{V}$ is applied across the terminals of the electrical device, the complex 3-D vector acoustic impedance $\tilde{Z}_a(\vec{r}, \omega)$ is a measure of the acoustical medium’s ability to **impede** the **flow** of a complex acoustic mass current $\tilde{I}_a(\vec{r}, \omega) = \tilde{J}_a(\vec{r}, \omega) \cdot \tilde{S}_\perp = \rho_0 \tilde{u}(\vec{r}, \omega) \cdot \tilde{S}_\perp (kg/s)$ for a complex over-pressure $\tilde{p}(\vec{r}, \omega)$ at point $\vec{r}$.

Similarly, just as complex scalar electrical **admittance** $\tilde{Y}_e = 1/\tilde{Z}_e$ is a measure of the **ease** with which an electrical device **admits** the **flow** of a complex scalar AC electrical current $\tilde{I}_e$ when a complex scalar AC voltage $\tilde{V}$ is applied across the terminals of the electrical device, the complex 3-D vector acoustic admittance $\tilde{Y}_a(\vec{r}, \omega) = 1/\tilde{Z}_a(\vec{r}, \omega)$ is a measure of the **ease** with which an acoustical medium’s **admits** the **flow** of a complex scalar acoustic mass current $\tilde{I}_a(\vec{r}, \omega) = \tilde{J}_a(\vec{r}, \omega) \cdot \tilde{S}_\perp = \rho_0 \tilde{u}(\vec{r}, \omega) \cdot \tilde{S}_\perp (kg/s)$ in the presence of a complex over-pressure $\tilde{p}(\vec{r}, \omega)$ at the point $\vec{r}$.

Another way to gain some physical insight into the nature of complex 3-D vector **specific** acoustic impedance $\tilde{Z}_a(\vec{r}, \omega) = \tilde{p}(\vec{r}, \omega)/\tilde{u}(\vec{r}, \omega)$ and complex 3-D vector **specific** acoustic admittance $\tilde{Y}_a(\vec{r}, \omega) = \tilde{u}(\vec{r}, \omega)/\tilde{p}(\vec{r}, \omega) = 1/\tilde{Z}_a(\vec{r}, \omega)$ of a medium associated with a harmonic sound field is to imagine a physical situation where we set the {magnitude} of the complex scalar over-pressure $\tilde{p}(\vec{r}, \omega)$ to be a constant/fixed value, e.g., $|\tilde{p}(\vec{r}, \omega)| = 1.0$ Pascal.

Then, for a harmonic sound field, if the complex 3-D vector **specific** acoustic impedance $\tilde{Z}_a(\vec{r}, \omega) = \tilde{p}(\vec{r}, \omega)/\tilde{u}(\vec{r}, \omega)$ at the point $\vec{r}$ happens to be very **high**, for a fixed complex scalar over-pressure $\tilde{p}(\vec{r}, \omega)$, this tells us that the complex 3-D vector particle velocity $\tilde{u}(\vec{r}, \omega)$ at that point must therefore be very **small**, and hence the corresponding complex 3-D vector acoustic mass current density $\tilde{J}_a(\vec{r}, \omega) = \rho_0 \tilde{u}(\vec{r}, \omega)$ at that point must also be very **small**.

Conversely, if for a harmonic sound field the complex 3-D vector **specific** acoustic impedance $\tilde{Z}_a(\vec{r}, \omega) = \tilde{p}(\vec{r}, \omega)/\tilde{u}(\vec{r}, \omega)$ at the point $\vec{r}$ happens to be very **low**, for a fixed complex scalar over-pressure $\tilde{p}(\vec{r}, \omega)$, this tells us that the complex 3-D vector particle velocity $\tilde{u}(\vec{r}, \omega)$ at that point must therefore be very **large**, and hence the corresponding complex 3-D vector acoustic mass current density $\tilde{J}_a(\vec{r}, \omega) = \rho_0 \tilde{u}(\vec{r}, \omega)$ at that point must also be very **large**.
For a harmonic sound field, if the complex 3-D vector specific admittance
\( \tilde{y}_a(r, \omega) = \tilde{u}(r, \omega)/\tilde{p}(r, \omega) = 1/\tilde{z}_a(r, \omega) \) at the point \( r \) happens to be very high, for a fixed complex scalar over-pressure \( \tilde{p}(r, \omega) \), this tells us that the complex 3-D vector particle velocity \( \tilde{u}(r, \omega) \) at that point must therefore be very large, and hence the corresponding complex 3-D vector acoustic mass current density \( \tilde{J}_a(r, \omega) = \rho_v \tilde{u}(r, \omega) \) at that point must also be very large.

Conversely, if for a harmonic sound field the complex 3-D vector specific acoustic admittance \( \tilde{y}_a(r, \omega) = \tilde{u}(r, \omega)/\tilde{p}(r, \omega) = 1/\tilde{z}_a(r, \omega) \) at the point \( r \) happens to be very low, for a fixed complex scalar over-pressure \( \tilde{p}(r, \omega) \), this tells us that the complex 3-D vector particle velocity \( \tilde{u}(r, \omega) \) at that point must therefore be very small, and hence the corresponding complex 3-D vector acoustic mass current density \( \tilde{J}_a(r, \omega) = \rho_v \tilde{u}(r, \omega) \) at that point must also be very small.

### The Real and Imaginary Components of Complex 3-D Vector Specific Acoustic Immittances:

As in the case for AC electrical circuits, the complex scalar electrical impedance \( \tilde{Z}_e \) and complex scalar electrical admittance \( \tilde{Y}_e = 1/\tilde{Z}_e \) can be written out explicitly in terms of their real and imaginary components:

\[
\tilde{Z}_e \equiv R_e + iX_e (\Omega) \quad \text{where} \quad R_e = \text{Re}\left\{\tilde{Z}_e\right\} \text{ is the resistance and } X_e = \text{Im}\left\{\tilde{Z}_e\right\} \text{ is the reactance.}
\]

\[
\tilde{Y}_e \equiv G_e + iB_e (\Omega^{-1}) \quad \text{where} \quad G_e = \text{Re}\left\{\tilde{Y}_e\right\} \text{ is the conductance and } B_e = \text{Im}\left\{\tilde{Y}_e\right\} \text{ is the susceptance.}
\]

Similarly, for the case a complex harmonic sound field \( \tilde{S}(\tilde{r}) \), the complex 3-D vector specific acoustic impedance \( \tilde{z}_a(\tilde{r}) \) and complex 3-D specific acoustic admittance \( \tilde{y}_a(\tilde{r}) = 1/\tilde{z}_a(\tilde{r}) \) can be written out explicitly in terms of their real and imaginary components:

\[
\tilde{z}_a(\tilde{r}, \omega) \equiv \tilde{r}_a(\tilde{r}, \omega) + i\tilde{\chi}_a(\tilde{r}, \omega) \quad (\Omega) \quad \text{where:}
\]

\[
\tilde{r}_a(\tilde{r}, \omega) = \text{Re}\left\{\tilde{z}_a(\tilde{r}, \omega)\right\} \text{ is the 3-D specific acoustic resistance at the point } \tilde{r} \text{ and:}
\]

\[
\tilde{\chi}_a(\tilde{r}, \omega) = \text{Im}\left\{\tilde{z}_a(\tilde{r}, \omega)\right\} \text{ is the 3-D specific acoustic reactance at the point } \tilde{r} .
\]

\[
\tilde{y}_a(\tilde{r}, \omega) \equiv \tilde{g}_a(\tilde{r}, \omega) + i\tilde{b}_a(\tilde{r}, \omega) \quad (\Omega^{-1}) \quad \text{where:}
\]

\[
\tilde{g}_a(\tilde{r}, \omega) = \text{Re}\left\{\tilde{y}_a(\tilde{r}, \omega)\right\} \text{ is the 3-D specific acoustic conductance at the point } \tilde{r} \text{ and:}
\]

\[
\tilde{b}_a(\tilde{r}, \omega) = \text{Im}\left\{\tilde{y}_a(\tilde{r}, \omega)\right\} \text{ is the 3-D specific acoustic susceptance at the point } \tilde{r} .
\]
For harmonic/single-frequency sound fields, we can obtain expressions for the real and imaginary parts of frequency-domain complex 3-D vector specific acoustic impedance $\tilde{z}_a(\tilde{r}, \omega)$ and admittance $\tilde{y}_a(\tilde{r}, \omega)$ in terms of the real and imaginary parts of complex scalar over-pressure $\tilde{p}(\tilde{r}, \omega)$ and complex 3-D vector particle velocity $\tilde{u}(\tilde{r}, \omega)$ from their respective definitions $\tilde{z}_a(\tilde{r}, \omega) = \tilde{p}(\tilde{r}, \omega)/\tilde{u}(\tilde{r}, \omega)$ and $\tilde{y}_a(\tilde{r}, \omega) = \tilde{u}(\tilde{r}, \omega)/\tilde{p}(\tilde{r}, \omega) = 1/\tilde{z}_a(\tilde{r}, \omega)$.

Suppressing the frequency-domain argument $(\tilde{r}, \omega)$ for notational clarity’s sake, and working with only one of the three vectorial components $k = x, y, or z$, for complex 3-D vector specific acoustic admittance:

$$\tilde{y}_a = y^r_a + iy^i_a = \frac{\tilde{u}_k}{\tilde{p}} = \frac{u_n + iu_{ni}}{p_r + ip_i} = \left(\frac{p_r - ip_i}{p_r + ip_i}\right) \left(\frac{p_r u_n + p_i u_{ni}}{|\tilde{p}|^2}\right) + i \left(\frac{p_r u_n - p_i u_{ni}}{|\tilde{p}|^2}\right)$$

Thus we see that for $k = x, y, or z$:

$$y^r_a = \text{Re}\{\tilde{y}_a\} = \frac{p_r u_n + p_i u_{ni}}{|\tilde{p}|^2} \quad \text{and} \quad y^i_a = \text{Im}\{\tilde{y}_a\} = \frac{p_r u_n - p_i u_{ni}}{|\tilde{p}|^2}$$

Likewise, for complex 3-D vector specific acoustic impedance:

$$\tilde{z}_a = z^r_a + iz^i_a = \frac{\tilde{p} \cdot \tilde{u}^*_k}{|\tilde{u}|^2} = \frac{(p_r + ip_i)(u_n + iu_{ni})}{|\tilde{u}|^2} = \frac{(p_r + ip_i)(u_n - iu_{ni})}{|\tilde{u}|^2} = \left(\frac{p_r u_n + p_i u_{ni}}{|\tilde{u}|^2}\right) + i \left(\frac{p_r u_n - p_i u_{ni}}{|\tilde{u}|^2}\right)$$

Thus, we see that for $k = x, y, or z$:

$$z^r_a = \text{Re}\{\tilde{z}_a\} = \frac{p_r u_n + p_i u_{ni}}{|\tilde{u}|^2} \quad \text{and} \quad z^i_a = \text{Im}\{\tilde{z}_a\} = \frac{p_r u_n - p_i u_{ni}}{|\tilde{u}|^2}$$

Noting that: $|\tilde{y}_a|^2 = \tilde{y}_a \cdot \tilde{y}^*_a = \frac{|\tilde{u}_k|^2}{|\tilde{p}|^2}$ and that: $|\tilde{z}_a|^2 = \tilde{z}_a \cdot \tilde{z}^*_a = \frac{|\tilde{p} \cdot \tilde{u}^*_k|^2}{|\tilde{u}|^2 \cdot |\tilde{p}|^2}$

We see that: $|\tilde{u}|^2 z^r_a = p_r u_n + p_i u_{ni} = |\tilde{p}|^2 y^r_a$ and that: $|\tilde{u}|^2 z^i_a = p_r u_n - p_i u_{ni} = -|\tilde{p}|^2 y^i_a$

or equivalently that: $z^r_a = |\tilde{z}_a|^2 y^r_a$ or $y^r_a = \frac{1}{|\tilde{z}_a|^2} z^r_a$ and that: $z^i_a = -|\tilde{z}_a|^2 y^i_a$ or $y^i_a = -\frac{1}{|\tilde{z}_a|^2} z^i_a$
Thus, we see that for a given \( k = x, y, \text{ or } z \) component of \( \mathbf{z}^\ast_a (\mathbf{r}, \omega) \):

\[
z^\ast_{a_k} = \text{Re}\{z_{a_k}\} = \frac{p_{uk_x} + p_{uk_y}}{|\mathbf{u}|^2} \quad \text{and} \quad z^j_{a_k} = \text{Im}\{z_{a_k}\} = \frac{p_{uk_x} - p_{uk_y}}{|\mathbf{u}|^2}
\]

and we see that for a given \( k = x, y, \text{ or } z \) component of \( \mathbf{y}^\ast_a (\mathbf{r}, \omega) \):

\[
y^r_{a_k} = \text{Re}\{y_{a_k}\} = \frac{p_{uk_x} + p_{uk_y}}{|\mathbf{p}|^2} \quad \text{and} \quad y^i_{a_k} = \text{Im}\{y_{a_k}\} = \frac{p_{uk_x} - p_{uk_y}}{|\mathbf{p}|^2}
\]

as well as: \( \mathbf{z}^\ast_{a_k} = \frac{\mathbf{y}^\ast_{a_k}}{|\mathbf{y}^\ast_{a_k}|^2} \) and: \( \mathbf{y}^\ast_{a_k} = \frac{\mathbf{z}^\ast_{a_k}}{|\mathbf{z}^\ast_{a_k}|^2} \) or equivalently: \( \mathbf{y}^\ast_{a_k} = |\mathbf{y}^\ast_{a_k}|^2 \mathbf{z}^\ast_{a_k} \) and: \( \mathbf{z}^\ast_{a_k} = |\mathbf{z}^\ast_{a_k}|^2 \mathbf{y}^\ast_{a_k} \).

It can be seen from these definitions that in general the individual vectorial components \( k = x, y, \text{ or } z \) that: \( \mathbf{z}_a (\mathbf{r}, \omega) \) and \( \mathbf{y}_a (\mathbf{r}, \omega) \) do not point in the same direction in space.

Since \( \mathbf{z}_a (\mathbf{r}, \omega) = \mathbf{p}_a (\mathbf{r}, \omega) / |\mathbf{u}(\mathbf{r}, \omega)| \), another useful relation is: \( \mathbf{z}_a (\mathbf{r}, \omega) \cdot \mathbf{u}(\mathbf{r}, \omega) = \mathbf{p}(\mathbf{r}, \omega) \):

\[
\mathbf{z}_a (\mathbf{r}, \omega) \cdot \mathbf{u}(\mathbf{r}, \omega) = \frac{\hat{\mathbf{p}}(\mathbf{r}, \omega)}{|\mathbf{u}(\mathbf{r}, \omega)|} \cdot |\mathbf{u}(\mathbf{r}, \omega)|^2 = \frac{\hat{\mathbf{p}}(\mathbf{r}, \omega) \mathbf{u}(\mathbf{r}, \omega)}{|\mathbf{u}(\mathbf{r}, \omega)|^2} = \mathbf{p}(\mathbf{r}, \omega)
\]

Similarly, since \( \mathbf{y}_a (\mathbf{r}, \omega) = \mathbf{u}(\mathbf{r}, \omega) / |\mathbf{p}(\mathbf{r}, \omega)| \), then: \( \mathbf{y}_a (\mathbf{r}, \omega) \cdot \mathbf{p}(\mathbf{r}, \omega) = \mathbf{u}(\mathbf{r}, \omega) \).

We can gain further/additional insight into the nature of complex \( \mathbf{z}_a (\mathbf{r}, \omega) \) and \( \mathbf{y}_a (\mathbf{r}, \omega) \) by writing our primary acoustic frequency-domain variables in complex polar notation form:

**Complex scalar pressure:**

\[\mathbf{p}(\mathbf{r}, \omega) = p_x (\mathbf{r}, \omega) + ip_y (\mathbf{r}, \omega) = |\mathbf{p}(\mathbf{r}, \omega)| e^{ip_y (\mathbf{r}, \omega)}\]

**Complex 3-D vector particle velocity:**

\[\mathbf{u}(\mathbf{r}, \omega) = u_x (\mathbf{r}, \omega) + iu_y (\mathbf{r}, \omega) = |\mathbf{u}(\mathbf{r}, \omega)| e^{ip_y (\mathbf{r}, \omega)} \cdot \hat{x} + |\mathbf{u}(\mathbf{r}, \omega)| e^{ip_x (\mathbf{r}, \omega)} \cdot \hat{y} + |\mathbf{u}(\mathbf{r}, \omega)| e^{ip_z (\mathbf{r}, \omega)} \cdot \hat{z}\]
Complex 3-D vector specific acoustic admittance:

\[
\tilde{y}_a(\tilde{r}, \omega) = \tilde{y}_x(\tilde{r}, \omega) + i\tilde{y}_y(\tilde{r}, \omega) = \left[ y_x(\tilde{r}, \omega) + iy_y(\tilde{r}, \omega) \right] \hat{x} + \left[ y_y(\tilde{r}, \omega) + iy_x(\tilde{r}, \omega) \right] \hat{y} + \left[ y_z(\tilde{r}, \omega) + iy_z(\tilde{r}, \omega) \right] \hat{z}
\]

Complex 3-D vector specific acoustic impedance:

\[
\tilde{z}_a(\tilde{r}, \omega) = \tilde{z}_x(\tilde{r}, \omega) + i\tilde{z}_y(\tilde{r}, \omega) = \left[ z_x(\tilde{r}, \omega) + iz_y(\tilde{r}, \omega) \right] \hat{x} + \left[ z_y(\tilde{r}, \omega) + iz_x(\tilde{r}, \omega) \right] \hat{y} + \left[ z_z(\tilde{r}, \omega) + iz_z(\tilde{r}, \omega) \right] \hat{z}
\]

Thus, for harmonic/single-frequency sound fields we see that for a given \( k = x, y, \) or \( z \) component of \( \tilde{y}_a(\tilde{r}, \omega) \), that:

\[
\tilde{y}_a(\tilde{r}, \omega) = \frac{\tilde{u}_k(\tilde{r}, \omega)}{\tilde{p}(\tilde{r}, \omega)} \Rightarrow \left| \tilde{u}_k \right| e^{i\varphi_{u_k}} = \frac{\left| \tilde{u}_k \right| e^{i\varphi_{u_k}}}{\left| \tilde{p} \right|} \cdot e^{i\varphi_{u_k}} = \left| \tilde{y}_a \right| e^{i\varphi_{u_k}} \Rightarrow \left| \tilde{y}_a \right| e^{-i\varphi_{u_k}}
\]

Similarly, for a given \( k = x, y, \) or \( z \) component of \( \tilde{z}_a(\tilde{r}, \omega) \):

\[
\tilde{z}_a(\tilde{r}, \omega) = \frac{\tilde{p}(\tilde{r}, \omega)\tilde{u}_k^*(\tilde{r}, \omega)}{\tilde{u}(\tilde{r}, \omega)^2} \Rightarrow \left| \tilde{z}_a \right| e^{i\varphi_{u_k}} = \frac{\left| \tilde{p} \right| e^{i\varphi_p} \left| \tilde{u}_k \right| e^{-i\varphi_{a_k}}}{\left| \tilde{u} \right|} \Rightarrow \left| \tilde{z}_a \right| e^{i\varphi_{u_k}} \Rightarrow \left| \tilde{z}_a \right| e^{i\varphi_{a_k}}
\]

We also see that for harmonic/single-frequency sound fields the \( z_k \)-phase: \( \varphi_{z_k} = \Delta \varphi_{p-a_k} \equiv \varphi_p - \varphi_{a_k} \)

whereas the \( y_k \)-phase: \( \varphi_{y_k} = \Delta \varphi_{a_k-p} \equiv \varphi_{a_k} - \varphi_p = -\left( \varphi_p - \varphi_{a_k} \right) = -\varphi_{z_k} \), in analogy to similar relations obtained e.g. for complex AC electrical circuits!

The phasor relation(s) in the complex plane for \( \tilde{p} = p_x + ip_y = |\tilde{p}| e^{i\varphi_p}, \tilde{u}_k = u_x + iu_y = |\tilde{u}_k| e^{i\varphi_{a_k}}, \tilde{z}_a = z_k^* + iz_k^i = |\tilde{z}_a| e^{i\varphi_{u_k}} \) and \( \tilde{y}_a = y_k^* + iy_k^i = |\tilde{y}_a| e^{i\varphi_{u_k}} \) are shown in the figure below, for the special/limiting case of \( \Delta \varphi_{p-a_k} \equiv \varphi_p - \varphi_{a_k} = \varphi_{z_k} = -\varphi_{y_k} = 90^{\circ} \), where the impedance phasor component \( \tilde{z}_a \) is back-to-back with the admittance phasor component \( \tilde{y}_a \) \( \{n.b. for the more general case where \( \Delta \varphi_{p-a_k} \equiv \varphi_p - \varphi_{a_k} = \varphi_{z_k} = -\varphi_{y_k} = 90^{\circ} \), then \( \tilde{z}_a \) and \( \tilde{y}_a \) are not back-to-back\}:
If we now take the cosine of the two phases $\varphi_{z_k}$ and $\varphi_{y_k}$:

$$\cos \varphi_{z_k} = \cos \Delta \varphi_{p-u_k} \equiv \cos(\varphi_p - \varphi_{u_k})$$

and:

$$\cos \varphi_{y_k} = \cos \Delta \varphi_{u_k-p} \equiv \cos(\varphi_{u_k} - \varphi_p) = \cos[-\varphi_{z_k}] = \cos \varphi_{z_k} \cos(x \text{ even fcn}(x))$$

We see that when: $\cos \varphi_{z_k} = \cos \varphi_{y_k} = +1$ that: $\Delta \varphi_{p-u_k} = -\Delta \varphi_{u_k-p} = 0^\circ$, i.e. that: $\varphi_p = \varphi_{u_k}$.

When: $\cos \varphi_{z_k} = \cos \varphi_{y_k} = 0$ that: $\Delta \varphi_{p-u_k} = -\Delta \varphi_{u_k-p} = \pm 90^\circ$, i.e. that: $\varphi_p = \varphi_{u_k} \pm 90^\circ$.

When: $\cos \varphi_{z_k} = \cos \varphi_{y_k} = -1$ that: $\Delta \varphi_{p-u_k} = -\Delta \varphi_{u_k-p} = \pm 180^\circ$, i.e. that: $\varphi_p = \varphi_{u_k} \pm 180^\circ$.

Summary of Various Frequency-Domain Sound Field Physical Quantities:

**Complex scalar pressure:**

$$\tilde{p}(\vec{r}, \omega) = p_r(\vec{r}, \omega) + ip_i(\vec{r}, \omega) = |\tilde{p}(\vec{r})|e^{ip_0(\vec{r}, \omega)}$$

**Complex 3-D vector particle displacement:**

$$\vec{\xi}(\vec{r}, \omega) = \xi_r(\vec{r}, \omega) + i\xi_i(\vec{r}, \omega)$$

$$= \left[\xi_{r_1}(\vec{r}, \omega) + i\xi_{i_1}(\vec{r}, \omega)\right]\hat{x} + \left[\xi_{r_2}(\vec{r}, \omega) + i\xi_{i_2}(\vec{r}, \omega)\right]\hat{y} + \left[\xi_{r_3}(\vec{r}, \omega) + i\xi_{i_3}(\vec{r}, \omega)\right]\hat{z}$$

$$= |\xi_{r_1}(\vec{r}, \omega)|e^{ip_{0_1}(\vec{r}, \omega)}\hat{x} + |\xi_{r_2}(\vec{r}, \omega)|e^{ip_{0_2}(\vec{r}, \omega)}\hat{y} + |\xi_{r_3}(\vec{r}, \omega)|e^{ip_{0_3}(\vec{r}, \omega)}\hat{z}$$

**Complex 3-D vector particle velocity:**

$$\vec{u}(\vec{r}, \omega) = u_r(\vec{r}, \omega) + iu_i(\vec{r}, \omega)$$

$$= \left[u_{r_1}(\vec{r}, \omega) + iu_{i_1}(\vec{r}, \omega)\right]\hat{x} + \left[u_{r_2}(\vec{r}, \omega) + iu_{i_2}(\vec{r}, \omega)\right]\hat{y} + \left[u_{r_3}(\vec{r}, \omega) + iu_{i_3}(\vec{r}, \omega)\right]\hat{z}$$

$$= |u_{r_1}(\vec{r}, \omega)|e^{ip_{0_1}(\vec{r}, \omega)}\hat{x} + |u_{r_2}(\vec{r}, \omega)|e^{ip_{0_2}(\vec{r}, \omega)}\hat{y} + |u_{r_3}(\vec{r}, \omega)|e^{ip_{0_3}(\vec{r}, \omega)}\hat{z}$$

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Complex 3-D vector particle acceleration:

\[
\vec{\ddot{a}}(\vec{r}, \omega) = a_x(\vec{r}, \omega) + ia_y(\vec{r}, \omega) + ia_z(\vec{r}, \omega)
\]

\[
= [a_x(\vec{r}, \omega) + ia_y(\vec{r}, \omega)]\hat{x} + [a_y(\vec{r}, \omega) + ia_z(\vec{r}, \omega)]\hat{y} + [a_y(\vec{r}, \omega) + ia_z(\vec{r}, \omega)]\hat{z}
\]

Complex 3-D vector specific acoustic admittance:

\[
\vec{y}_a(\vec{r}, \omega) = y_x(\vec{r}, \omega) + iy_y(\vec{r}, \omega)
\]

\[
= [y_x(\vec{r}, \omega) + iy_y(\vec{r}, \omega)]\hat{x} + [y_y(\vec{r}, \omega) + iy_y(\vec{r}, \omega)]\hat{y} + [y_y(\vec{r}, \omega) + iy_y(\vec{r}, \omega)]\hat{z}
\]

Complex 3-D vector specific acoustic impedance:

\[
\vec{z}_a(\vec{r}, \omega) = z_x(\vec{r}, \omega) + iz_y(\vec{r}, \omega)
\]

\[
= [z_x(\vec{r}, \omega) + iz_y(\vec{r}, \omega)]\hat{x} + [z_y(\vec{r}, \omega) + iz_y(\vec{r}, \omega)]\hat{y} + [z_y(\vec{r}, \omega) + iz_y(\vec{r}, \omega)]\hat{z}
\]

For “everyday” harmonic/single-frequency sound fields, if the 3-D vector complex frequency-domain particle velocity amplitude \(\vec{u}(\vec{r}, \omega)\) is known/measured, then since the 3-D vector complex time-domain particle velocity \(\vec{u}(\vec{r}, t) = \vec{u}(\vec{r}, \omega) \cdot e^{i\omega t}\), and the 3-D vector complex time-domain particle displacement \(\vec{z}_a(\vec{r}, t) = \vec{z}_a(\vec{r}, \omega) \cdot e^{i\omega t}\), where: \(\vec{z}_a(\vec{r}, \omega)\) is the 3-D vector complex frequency-domain particle displacement amplitude, and since \(\vec{u}(\vec{r}, t) = \partial \vec{z}_a(\vec{r}, t) / \partial t\), then:

\[
\vec{z}_a(\vec{r}, t) = \int \vec{u}(\vec{r}, t) dt = \int \vec{u}(\vec{r}, \omega) \cdot e^{i\omega t} dt = \vec{u}(\vec{r}, \omega) \int e^{i\omega t} dt = \frac{1}{i\omega} \vec{u}(\vec{r}, \omega) \cdot e^{i\omega t}
\]

But since:

\[
\vec{z}_a(\vec{r}, t) = \vec{z}_a(\vec{r}, \omega) \cdot e^{i\omega t}, \quad \text{we see that:} \quad \vec{z}_a(\vec{r}, \omega) = \frac{1}{i\omega} \vec{u}(\vec{r}, \omega) = -\frac{1}{\omega} \vec{u}(\vec{r}, \omega)
\]

Likewise, since:

\[
\vec{a}(\vec{r}, t) = \frac{\partial \vec{u}(\vec{r}, t)}{\partial t} = \frac{\partial \vec{u}(\vec{r}, \omega)}{\partial t} \cdot e^{i\omega t} \Rightarrow \vec{a}(\vec{r}, \omega) \cdot \frac{\partial e^{i\omega t}}{\partial t} = i\omega \cdot \vec{u}(\vec{r}, \omega) \cdot e^{i\omega t}
\]

But since:

\[
\vec{a}(\vec{r}, t) = \vec{a}(\vec{r}, \omega) \cdot e^{i\omega t}, \quad \text{we also see that:} \quad \vec{a}(\vec{r}, \omega) = i\omega \cdot \vec{u}(\vec{r}, \omega)
\]
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