Calculation of Inharmonicities in Musical Instruments

As we have discussed in the POM lectures, real 1-dimensional musical instruments do not have a perfect/ideal harmonic sequence of overtones – i.e. \( f_n^{\text{thy}} = n \cdot f_1^{\text{thy}} \), \( n = 1, 2, 3, \ldots \), in contrast to the predictions of the perfect/ideal “simple” first-order theory of how the instrument works – such first-order theories neglect/ignore higher-order real-world effects such as finite string stiffness, viscous damping/dissipative effects that do indeed change/shift/perturb the resonant frequencies from their “ideal” “simple” first-order theory predictions.

For any given 1-dimensional musical instrument, if we measure the frequencies of the individual harmonics of the instrument \( f_n^{\text{expt}} \), we can compare them with the perfect/ideal “simple” first-order theory prediction \( f_n^{\text{thy}} = n \cdot f_1^{\text{thy}} \), \( n = 1, 2, 3, \ldots \) of the harmonic sequence of overtones to see how close to perfect/ideal the musical instrument actually is. In general, the experimental results will be close to, but not precisely identical with the simple perfect/ideal theory predictions. The deviation of each harmonic \( f_n^{\text{expt}} \) from the perfect/ideal prediction \( f_n^{\text{thy}} = n \cdot f_1^{\text{thy}} \), \( n = 1, 2, 3, \ldots \) is a measure of the inharmonicity of the musical instrument.

1.) Stringed Instrument Inharmonicities:

For stringed instruments, the fundamental frequency \( f_1^{\text{expt}} \) of an open vibrating string is not significantly perturbed from its perfect/ideal theory value of \( f_1^{\text{thy}} = v/\lambda_1 = 2 \sqrt{T/\mu}/L \), thus we can use it as a reference for the higher harmonics \( f_n^{\text{thy}} = n \cdot f_1^{\text{thy}} \), \( n = 1, 2, 3, \ldots \).

We can calculate the frequency ratios \( R_n^{\text{expt}} \equiv \left( f_n^{\text{expt}} / f_1^{\text{expt}} \right) = n_{\text{expt}} \) \( \neq n = 1, 2, 3, \ldots \) and compare these results to the “simple” perfect/ideal theory predictions \( R_n^{\text{thy}} \equiv \left( f_n^{\text{thy}} / f_1^{\text{thy}} \right) = n \), \( n = 1, 2, 3, \ldots \).

The measured frequency ratios \( R_n^{\text{expt}} \equiv \left( f_n^{\text{expt}} / f_1^{\text{expt}} \right) = n_{\text{expt}} \) should be close to, but will not be precisely equal to \( n \), \( n = 1, 2, 3, \ldots \) for stringed instruments, e.g. due to finite string stiffness and, to a lesser extent, due to the effect(s) of viscous damping/dissipation effects of the surrounding air in proximity of the vibrating string.

The deviation of a given measured frequency ratio \( R_n^{\text{expt}} = \left( f_n^{\text{expt}} / f_1^{\text{expt}} \right) \) from its perfect/ideal theory value \( R_n^{\text{thy}} \equiv \left( f_n^{\text{thy}} / f_1^{\text{thy}} \right) = n \), \( n = 1, 2, 3, \ldots \) is given by the experiment vs. theory difference between these two quantities:

\[
\Delta R_n^{\text{expt-thy}} \equiv R_n^{\text{expt}} - R_n^{\text{thy}} = \left( f_n^{\text{expt}} / f_1^{\text{expt}} \right) - \left( f_n^{\text{thy}} / f_1^{\text{thy}} \right) = n_{\text{expt}} - n
\]

If we then normalize the above expression to the theory ratio \( R_n^{\text{thy}} \equiv \left( f_n^{\text{thy}} / f_1^{\text{thy}} \right) = n \), \( n = 1, 2, 3, \ldots \) then we obtain the fractional deviation of a given measured frequency ratio \( R_n^{\text{expt}} \equiv \left( f_n^{\text{expt}} / f_1^{\text{expt}} \right) \) from its perfect/ideal theory value \( R_n^{\text{thy}} \equiv \left( f_n^{\text{thy}} / f_1^{\text{thy}} \right) = n \), \( n = 1, 2, 3, \ldots \):
\[ \frac{\Delta R_{n, \text{expt-thy}}}{R_{n, \text{thy}}} \equiv \frac{R_{n, \text{expt}} - R_{n, \text{thy}}}{R_{n, \text{thy}}} = \frac{\left( \frac{f_{n, \text{expt}}}{f_{1, \text{expt}}} \right) - \left( \frac{f_{n, \text{thy}}}{f_{1, \text{thy}}} \right)}{f_{n, \text{thy}}} = \frac{n_{\text{expt}} - n}{n} = \Delta n_{\text{expt-thy}} \]

Next, if we multiply this expression by the theory value of the frequency of this harmonic \( f_{n, \text{thy}} \), then this is equal to the shift/departure of this harmonic’s frequency from its perfect/ideal theory value \( \Delta f_{n, \text{expt-thy}} \) (in Hz):

\[ \Delta f_{n, \text{expt-thy}} \ (Hz) = f_{n, \text{thy}} \left( \frac{\Delta R_{n, \text{expt-thy}}}{R_{n, \text{thy}}} \right) = f_{n, \text{thy}} \left( \frac{\left( \frac{f_{n, \text{expt}}}{f_{1, \text{expt}}} \right) - \left( \frac{f_{n, \text{thy}}}{f_{1, \text{thy}}} \right)}{f_{n, \text{thy}}} \right) = f_{n, \text{thy}} \left( \frac{n_{\text{expt}} - n}{n} \right) = f_{n, \text{thy}} \left( \frac{\Delta n_{\text{expt-thy}}}{n} \right) \]

Stated another way:

\[ \left( \frac{\Delta f_{n, \text{expt-thy}}}{f_{n, \text{thy}}} \right) = \left( \frac{\Delta R_{n, \text{expt-thy}}}{R_{n, \text{thy}}} \right) = \left( \frac{R_{n, \text{expt}} - R_{n, \text{thy}}}{R_{n, \text{thy}}} \right) = \left( \frac{\left( \frac{f_{n, \text{expt}}}{f_{1, \text{expt}}} \right) - \left( \frac{f_{n, \text{thy}}}{f_{1, \text{thy}}} \right)}{f_{n, \text{thy}}} \right) = \left( \frac{n_{\text{expt}} - n}{n} \right) = \left( \frac{\Delta n_{\text{expt-thy}}}{n} \right) \]

Then the \% deviations \( \Delta_{n, \text{expt-thy}} \) (\%) of the actual/measured harmonics of a stringed instrument from their perfect/ideal theory values are obtained by multiplying the above expression by 100%:

\[ \Delta_{n, \text{expt-thy}} \ (%) = 100 \left( \frac{\Delta f_{n, \text{expt-thy}}}{f_{n, \text{thy}}} \right) = 100 \left( \frac{\Delta R_{n, \text{expt-thy}}}{R_{n, \text{thy}}} \right) = 100 \left( \frac{R_{n, \text{expt}} - R_{n, \text{thy}}}{R_{n, \text{thy}}} \right) \]

\[ = 100 \left[ \left( \frac{f_{n, \text{expt}}}{f_{1, \text{expt}}} \right) - \left( \frac{f_{n, \text{thy}}}{f_{1, \text{thy}}} \right) \right] = 100 \left( \frac{n_{\text{expt}} - n}{n} \right) = 100 \left( \frac{\Delta n_{\text{expt-thy}}}{n} \right) \]

Now one semi-tone in the tempered scale is equal to 100 cents, and since there are 12 notes in the tempered scale, then one octave = 1200 cents. The frequency ratio between two adjacent notes (i.e. a semitone) in the tempered scale is \( a^{\frac{1}{12}} = 2^{\frac{1}{12}} = 1.0594631 \).

Thus, one semitone low from the frequency \( f_{n, \text{thy}} \) is given by the formula:

\[ f_{\text{semi-tone \ low}} = f_{n, \text{thy}} / a^{\frac{1}{12}} = f_{n, \text{thy}} / 2^{\frac{1}{12}} = f_{n, \text{thy}} / 1.0594631 = 0.943874 f_{n, \text{thy}} \]

and one semitone high from the frequency \( f_{n, \text{thy}} \) is given by the formula:

\[ f_{\text{semi-tone \ high}} = a^{\frac{1}{12}} f_{n, \text{thy}} = 2^{\frac{1}{12}} f_{n, \text{thy}} = 1.0594631 f_{n, \text{thy}} \]
The frequency difference $\Delta f_{\text{low}}^{\text{semitone}}$ for one semitone low from $f_n^{\text{thy}}$ is given by:

$$\Delta f_{\text{low}}^{\text{semitone}} = f_n^{\text{thy}} - f_{n-1}^{\text{thy}} = f_n^{\text{thy}} - f_n^{\text{thy}}/\sqrt{2} = f_n^{\text{thy}}\left(1-1/\sqrt{2}\right) = f_n^{\text{thy}}\left(1-1/2^{1/2}\right)$$

$$= f_n^{\text{thy}}(1-0.943874)$$

$$= 0.0561257 f_n^{\text{thy}} \text{ (Hz)}$$

The frequency difference $\Delta f_{\text{high}}^{\text{semitone}}$ for one semitone high from $f_n^{\text{thy}}$ is given by:

$$\Delta f_{\text{high}}^{\text{semitone}} = f_n^{\text{thy}} - f_n^{\text{thy}}/\sqrt{2} = f_n^{\text{thy}} - f_n^{\text{thy}}(\sqrt{2} - 1) = f_n^{\text{thy}}(2^{1/2} - 1)$$

$$= 0.0594630 f_n^{\text{thy}} \text{ (Hz)}$$

Notice that $\left(\Delta f_{\text{low}}^{\text{semitone}} = 0.0561257 f_n^{\text{thy}}\right) \neq \left(\Delta f_{\text{high}}^{\text{semitone}} = 0.0594630 f_n^{\text{thy}}\right)$. They’re close, but they are not precisely equal to each other.

Suppose the frequency of a note on the musical scale, e.g. A5 is $f_n^{\text{thy}} = 880.00 \text{ Hz}$. The frequency of A5 (one semitone low) is 830.61 Hz, differing from A5 by 49.39 Hz. The frequency of B5 (one semitone high) is 932.33 Hz, differing from A5 by 52.33 Hz.

In order to express our measured experimental vs. theory frequency differences $\Delta f_n^{\text{exp-\text{thy}}}$ in cents, we need to a.) first normalize $\Delta f_n^{\text{exp-\text{thy}}}$ to $\Delta f_{\text{low}}^{\text{semitone}}$ (if $\Delta f_n^{\text{exp-\text{thy}}} < 0$) or normalize $\Delta f_n^{\text{exp-\text{thy}}}$ to $\Delta f_{\text{high}}^{\text{semitone}}$ (if $\Delta f_n^{\text{exp-\text{thy}}} > 0$), then b.) multiply each of these ratios by 100 cents:

If $\Delta f_n^{\text{exp-\text{thy}}} < 0$: $\# \text{ cents low} = 100 \left(\frac{\Delta f_n^{\text{exp-\text{thy}}}}{\Delta f_{\text{low}}^{\text{semitone}}}\right) \text{ cents}$

If $\Delta f_n^{\text{exp-\text{thy}}} > 0$: $\# \text{ cents high} = 100 \left(\frac{\Delta f_n^{\text{exp-\text{thy}}}}{\Delta f_{\text{high}}^{\text{semitone}}}\right) \text{ cents}$

Then, inserting the formulas for $\Delta f_n^{\text{exp-\text{thy}}}$, $\Delta f_{\text{low}}^{\text{semitone}}$ and $\Delta f_{\text{high}}^{\text{semitone}}$ in these expressions, we have:

If $\Delta f_n^{\text{exp-\text{thy}}} < 0$: $\# \text{ cents low} = 100 \left(\frac{f_n^{\text{thy}}}{f_n^{\text{thy}}(1-1/2^{1/2})}\right) \left(\frac{n_{\text{exp}} - n}{n}\right) = 100 \left(\frac{n_{\text{exp}} - n}{n}\right) \left(\frac{1}{1-1/2^{1/2}}\right) \left(\frac{1}{1-1/2^{1/2}}\right) \text{ cents}$

If $\Delta f_n^{\text{exp-\text{thy}}} > 0$: $\# \text{ cents high} = 100 \left(\frac{f_n^{\text{thy}}}{f_n^{\text{thy}}(2^{1/2} - 1)}\right) \left(\frac{n_{\text{exp}} - n}{n}\right) = 100 \left(\frac{n_{\text{exp}} - n}{n}\right) \left(\frac{1}{2^{1/2} - 1}\right) \left(\frac{1}{2^{1/2} - 1}\right) \text{ cents}$

where: $n_{\text{exp}} \equiv \left(\frac{f_n^{\text{exp}}}{f_{n-1}^{\text{exp}}}\right)$ and: $n \equiv \left(\frac{f_n^{\text{thy}}}{f_{n-1}^{\text{thy}}}\right)$, $n = 1, 2, 3, ...$
Numerically, these formulas, for **stringed instrument inharmonicities** are:

If $\Delta f_n^{\text{expt-thy}} < 0$:

$$
\text{# cents low} = \frac{100}{\left(1 - 1/2^{\frac{1}{2}}\right)} \left(\frac{n_{\text{expt}}}{n} - 1\right) = 100 \times \frac{0.0561257}{\left(\frac{n_{\text{expt}}}{n} - 1\right)} = 1781.715 \left(\frac{n_{\text{expt}}}{n} - 1\right) \text{ cents}
$$

If $\Delta f_n^{\text{expt-thy}} > 0$:

$$
\text{# cents high} = \frac{100}{\left(2^{\frac{1}{2}} - 1\right)} \left(\frac{n_{\text{expt}}}{n} - 1\right) = 100 \times \frac{0.0594630}{\left(\frac{n_{\text{expt}}}{n} - 1\right)} = 1681.718 \left(\frac{n_{\text{expt}}}{n} - 1\right) \text{ cents}
$$

where: $n_{\text{expt}} \equiv \left(\frac{f_{n \text{expt}}}{f_1 \text{expt}}\right)$ and: $n \equiv \left(\frac{f_n \text{thy}}{f_1 \text{thy}}\right), \ n = 1, 2, 3, ...$

II.) **Wind/Brass Lip-Reed Instrument Inharmonicities:**

For *wind/brass lip-reed* instruments, the **fundamental** frequency $f_1 \text{expt}$ of an open vibrating string is significantly perturbed from its **perfect/ideal** theory value of $f_1 \text{thy} = v/\lambda_1$, because of a variety of higher-order effects not taken into account in the simple perfect/ideal theory – e.g. frequency-dependent wavelength effects at the bell end of the instrument; the (longitudinal) speed of sound propagation in the confined/internal space of the instrument is also formally frequency dependent, i.e. $v = v(f)$ and especially so a lower frequencies, due to frequency-dependent viscous damping/dissipation effects. In wind/brass lip-reed instruments, the fundamental frequency $f_1 \text{expt}$ (aka the “pedal note” tends to be systematically pulled low, and significantly so. Thus, we **cannot** use the fundamental as a reference for the higher harmonics $f_n \text{expt} = n \cdot f_1 \text{expt}, \ n = 1, 2, 3, ...$. The pedal note/fundamental is also extremely difficult to play on wind/brass lip-reed instruments, and also not with any great accuracy. The second harmonic of wind/brass lip-reed instruments is much less adversely affected than the fundamental (= the first harmonic), and also easy to play, accurately so. Hence we can/will use the second harmonic as a \textbf{reference} for the higher harmonics in wind/brass lip-reed instruments.

We can then calculate the **frequency ratios** $R_n^{\text{expt}} \equiv \left(\frac{f_n \text{expt}}{f_2 \text{expt}}\right) = n_{\text{expt}}/2$ ($\neq n = 1, 2, 3, ...$) and compare these results to the “simple” **perfect/ideal** theory predictions $R_n^{\text{thy}} \equiv \left(\frac{f_n \text{thy}}{f_2 \text{thy}}\right) = n/2, \ n = 1, 2, 3, ...$

The measured frequency ratios $R_n^{\text{expt}} \equiv \left(\frac{f_n \text{expt}}{f_2 \text{expt}}\right) = n_{\text{expt}}/2$ should be close to, but will not be precisely equal to $n/2, \ n = 1, 2, 3, ...$ for wind/brass lip-reed instruments, due to the effect(s) of viscous damping/dissipation effects of the air inside the bore of and in contact with the inside surface(s) of the instrument.
The deviation of a given measured frequency ratio $R^\text{expt}_n \equiv (f_{n}^{\text{expt}} / f_2^{\text{expt}})$ from its perfect/ideal “simple” theory value $R^\text{thy}_n \equiv (f_{n}^{\text{thy}} / f_2^{\text{thy}}) = n/2$, \( n = 1, 2, 3, \ldots \) is given by the experiment vs. theory difference between these two quantities:

$$\Delta R^\text{expt-thy}_n \equiv R^\text{expt}_n - R^\text{thy}_n = \left( \frac{f_{n}^{\text{expt}}}{f_2^{\text{expt}}} \right) - \left( \frac{f_{n}^{\text{thy}}}{f_2^{\text{thy}}} \right) = \frac{n_{\text{expt}}}{2} - \frac{n}{2} = \frac{\Delta n_{\text{expt-thy}}}{2}$$

If we then normalize the above expression to the theory ratio $R^\text{thy}_n \equiv (f_{n}^{\text{thy}} / f_2^{\text{thy}}) = n/2$, \( n = 1, 2, 3, \ldots \) then we obtain the fractional deviation of a given measured frequency ratio $R^\text{expt}_n \equiv (f_{n}^{\text{expt}} / f_2^{\text{expt}})$ from its perfect/ideal theory value $R^\text{thy}_n \equiv (f_{n}^{\text{thy}} / f_2^{\text{thy}})$:

$$\frac{\Delta R^\text{expt-thy}_n}{R^\text{thy}_n} = \frac{R^\text{expt}_n - R^\text{thy}_n}{R^\text{thy}_n} = \left( \frac{f_{n}^{\text{expt}}}{f_2^{\text{expt}}} \right) - \left( \frac{f_{n}^{\text{thy}}}{f_2^{\text{thy}}} \right) = \frac{n_{\text{expt}} - n}{n} = \frac{\Delta n_{\text{expt-thy}}}{n}$$

Next, if we multiply this expression by the theory value of the frequency of this harmonic $f_{n}^{\text{thy}}$, then this is equal to the shift/departure of this harmonic’s frequency from its perfect/ideal theory value $\Delta f_{n}^{\text{expt-thy}}$ (in Hz):

$$\Delta f_{n}^{\text{expt-thy}} (Hz) = f_{n}^{\text{thy}} \left( \frac{\Delta R^\text{expt-thy}_n}{R^\text{thy}_n} \right) = f_{n}^{\text{thy}} \left( \frac{n_{\text{expt}} - n}{n} \right) = f_{n}^{\text{thy}} \left( \frac{\Delta n_{\text{expt-thy}}}{n} \right)$$

Stated another way:

$$\left( \frac{\Delta f_{n}^{\text{expt-thy}}}{f_{n}^{\text{thy}}} \right) = \left( \frac{\Delta R^\text{expt-thy}_n}{R^\text{thy}_n} \right) = \left( \frac{R^\text{expt}_n - R^\text{thy}_n}{R^\text{thy}_n} \right) = \left( \frac{f_{n}^{\text{expt}}}{f_2^{\text{expt}}} \right) - \left( \frac{f_{n}^{\text{thy}}}{f_2^{\text{thy}}} \right) = \left( \frac{n_{\text{expt}} - n}{n} \right) = \left( \frac{\Delta n_{\text{expt-thy}}}{n} \right)$$

Then the % deviations $\Delta n_{\text{expt-thy}} (%)$ of the actual/measured harmonics of a stringed instrument from their perfect/ideal theory values are obtained by multiplying the above expression by 100%:

$$\Delta n_{\text{expt-thy}} (%) \equiv 100 \left( \frac{\Delta f_{n}^{\text{expt-thy}}}{f_{n}^{\text{thy}}} \right) = 100 \left( \frac{\Delta R^\text{expt-thy}_n}{R^\text{thy}_n} \right) = 100 \left( \frac{R^\text{expt}_n - R^\text{thy}_n}{R^\text{thy}_n} \right)$$

$$= 100 \left[ \left( \frac{f_{n}^{\text{expt}}}{f_2^{\text{expt}}} \right) - \left( \frac{f_{n}^{\text{thy}}}{f_2^{\text{thy}}} \right) \right] = 100 \left( \frac{n_{\text{expt}} - n}{n} \right) = 100 \left( \frac{\Delta n_{\text{expt-thy}}}{n} \right)$$
Now one semi-tone in the **tempered scale** is equal to **100 cents**, and since there are 12 notes in the **tempered scale**, then one octave = 1200 cents. The frequency ratio between two adjacent notes (i.e. a semitone) in the tempered scale is $a^{1/12} = 2^{1/12} = 1.0594631$.

Thus, one semitone **low** from the frequency $f_n^{\text{thy}}$ is given by the formula:

$$f_{n}^{\text{semitone low}} = f_n^{\text{thy}} / a^{1/12} = f_n^{\text{thy}} / 2^{1/12} = f_n^{\text{thy}} / 1.0594631 = 0.943874f_n^{\text{thy}}$$

and one semitone **high** from the frequency $f_n^{\text{thy}}$ is given by the formula:

$$f_{n}^{\text{semitone high}} = a^{1/12} f_n^{\text{thy}} = 2^{1/12} f_n^{\text{thy}} = 1.0594631f_n^{\text{thy}}$$

The frequency **difference** $\Delta f_{n}^{\text{semitone}}$ for one semitone **low** from $f_n^{\text{thy}}$ is given by:

$$\Delta f_{n}^{\text{semitone low}} = f_n^{\text{thy}} - f_{n}^{\text{semitone low}} = f_n^{\text{thy}} - f_n^{\text{thy}} / a^{1/12} = f_n^{\text{thy}} (1 - 1/a^{1/12}) = f_n^{\text{thy}} (1 - 1/2^{1/12})$$

$$= f_n^{\text{thy}} (1 - 0.943874)$$

$$= 0.0561257f_n^{\text{thy}} \text{ (Hz)}$$

The frequency **difference** $\Delta f_{n}^{\text{semitone}}$ for one semitone **high** from $f_n^{\text{thy}}$ is given by:

$$\Delta f_{n}^{\text{semitone high}} = f_{n}^{\text{semitone high}} - f_n^{\text{thy}} = a^{1/12} f_n^{\text{thy}} - f_n^{\text{thy}} = f_n^{\text{thy}} (a^{1/12} - 1) = f_n^{\text{thy}} (2^{1/12} - 1)$$

$$= f_n^{\text{thy}} (1.0594630 - 1)$$

$$= 0.0594630f_n^{\text{thy}} \text{ (Hz)}$$

Notice that $(\Delta f_{n}^{\text{semitone low}} = 0.0561257f_n^{\text{thy}}) \neq (\Delta f_{n}^{\text{semitone high}} = 0.0594630f_n^{\text{thy}})$. They’re close, but they are not **precisely** equal to each other.

Suppose the frequency of a note on the musical scale, e.g. A5 is $f_n^{\text{thy}} = 880.00$ Hz.

The frequency of A♭5 (one semitone **low**) is 830.61 Hz, differing from A5 by 49.39 Hz.

The frequency of B♭5 (one semitone **high**) is 932.33 Hz, differing from A5 by 52.33 Hz.

In order to express our **measured experimental** vs. **theory** frequency **differences** $\Delta f_n^{\text{exp-thy}}$ in **cents**, we need to a.) first **normalize** $\Delta f_n^{\text{exp-thy}}$ to $\Delta f_{n}^{\text{semitone low}}$ (if $\Delta f_n^{\text{exp-thy}} < 0$) or **normalize** $\Delta f_n^{\text{exp-thy}}$ to $\Delta f_{n}^{\text{semitone high}}$ (if $\Delta f_n^{\text{exp-thy}} > 0$), then b.) **multiply each** of these **ratios** by **100 cents**:

If $\Delta f_n^{\text{exp-thy}} < 0$: # cents **low** = $100 \left( \frac{\Delta f_n^{\text{exp-thy}}}{\Delta f_{n}^{\text{semitone low}}} \right) \text{ cents}$

If $\Delta f_n^{\text{exp-thy}} > 0$: # cents **high** = $100 \left( \frac{\Delta f_n^{\text{exp-thy}}}{\Delta f_{n}^{\text{semitone high}}} \right) \text{ cents}$
Then, inserting the formulas for \( \Delta f_n^{\text{expt-thy}} \), \( \Delta f_{\text{low}}^{\text{semitone}} \) and \( \Delta f_{\text{high}}^{\text{semitone}} \) in these expressions, we have:

<table>
<thead>
<tr>
<th>Equation</th>
<th>Condition</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>If ( \Delta f_n^{\text{expt-thy}} &lt; 0 ):</td>
<td># cents low = ( 100 \left( \frac{f_n^{\text{expt}}}{f_n^{\text{th}} - \frac{1}{2^{1/2}}} \right) \left( \frac{n_{\text{expt}} - n}{n} \right) = \frac{100}{1 - \frac{1}{2^{1/2}}} \left( \frac{n_{\text{expt}} - 1}{n} \right) ) cents</td>
<td></td>
</tr>
<tr>
<td>If ( \Delta f_n^{\text{expt-thy}} &gt; 0 ):</td>
<td># cents high = ( 100 \left( \frac{f_n^{\text{expt}}}{f_n^{\text{th}} + \frac{1}{2^{1/2}}} \right) \left( \frac{n_{\text{expt}} - n}{n} \right) = \frac{100}{\frac{1}{2^{1/2}} - 1} \left( \frac{n_{\text{expt}} - 1}{n} \right) ) cents</td>
<td></td>
</tr>
</tbody>
</table>

where: \( n_{\text{expt}} \equiv 2 \left( \frac{f_n^{\text{expt}}}{f_2^{\text{expt}}} \right) \) and: \( n \equiv 2 \left( \frac{f_n^{\text{th}}}{f_2^{\text{th}}} \right), \ n = 1, 2, 3,... \)

Numerically, these formulas, for wind/brass lip-reed instrument inharmonicities are:

If \( \Delta f_n^{\text{expt-thy}} < 0 \):

\[
\# \text{ cents low} = \frac{100}{1 - \frac{1}{2^{1/2}}} \left( \frac{n_{\text{expt}} - 1}{n} \right) = \frac{100}{0.0561257} \left( \frac{n_{\text{expt}} - 1}{n} \right) = 1781.715 \left( \frac{n_{\text{expt}} - 1}{n} \right) \text{ cents}
\]

If \( \Delta f_n^{\text{expt-thy}} > 0 \):

\[
\# \text{ cents high} = \frac{100}{\frac{1}{2^{1/2}} - 1} \left( \frac{n_{\text{expt}} - 1}{n} \right) = \frac{100}{0.0594630} \left( \frac{n_{\text{expt}} - 1}{n} \right) = 1681.718 \left( \frac{n_{\text{expt}} - 1}{n} \right) \text{ cents}
\]

where: \( n_{\text{expt}} \equiv 2 \left( \frac{f_n^{\text{expt}}}{f_2^{\text{expt}}} \right) \) and: \( n \equiv 2 \left( \frac{f_n^{\text{th}}}{f_2^{\text{th}}} \right), \ n = 1, 2, 3,... \)
Legal Disclaimer and Copyright Notice:

Legal Disclaimer:

The author specifically disclaims legal responsibility for any loss of profit, or any consequential, incidental, and/or other damages resulting from the mis-use of information contained in this document. The author has made every effort possible to ensure that the information contained in this document is factually and technically accurate and correct.

Copyright Notice:

The contents of this document are protected under both United States of America and International Copyright Laws. No portion of this document may be reproduced in any manner for commercial use without prior written permission from the author of this document. The author grants permission for the use of information contained in this document for private, non-commercial purposes only.