Introduction

- Uncertainties exist in all experiments
- The final goal of any experiment is to obtain *reproducible* results. Knowing errors and uncertainties is an essential part for ensuring reproducibility.
- To know the uncertainties we use two approaches:
  1. Repeat each measurement many times and determine how well the result reproduces itself.
  2. Measure the quantity of interest using a different method. The result has to be independently of the measurement technique. Thus systematic errors can be revealed.
- Presenting the result of your experiment: Use the right number of significant digits, in agreement with the estimated uncertainty.
Outline of the lecture

- Errors and uncertainties
- The reading error
- Accuracy and precession
- Systematic and statistical errors
- Fitting errors
- Presentation of the results
Errors (uncertainties)

Result of measurement

\[ X_{\text{meas}} = X_{\text{true}} + e_s + e_r \]

Correct value

Systematic error

Random error

\[ e_s = 0 \]
Systematic vs. Statistical Uncertainties

• Systematic uncertainty
  – Uncertainties associated with imperfect knowledge of measurement apparatus, other physical quantities needed for the measurement, or the physical model used to interpret the data.
  – Generally correlated between measurements. Cannot be reduced by multiple measurements.
  – Better calibration, or measurements employing different techniques or methods can reduce the uncertainty.

• Statistical Uncertainty
  – Uncertainties due to stochastic fluctuations
  – Generally there is no correlation between successive measurements.
  – Multiple measurements can be used to reduce this uncertainty.
The Difference Between Systematic & Random Errors

• Random error describes errors that fluctuate due to the unpredictability or uncertainty inherent in your measuring process, or the variation in the quantity you’re trying to measure. Such errors can be reduced by repeating the measurement and averaging the results.

• A systematic error is one that results from a persistent issue and leads to a consistent error in your measurements. For example, if your measuring tape has been stretched out, your results will always be lower than the true value. Similarly, if you’re using scales that haven’t been set to zero beforehand, there will be a systematic error resulting from the mistake in the calibration. Such errors cannot be reduced simply by repeating the measurement and averaging the results. Such errors can be reduced by analyzing the instrument(s) used for the measurement and by using different instruments.
The standard uncertainty $u(y)$ of a measurement result $y$ is the estimated standard deviation of $y$.

The relative standard uncertainty $ur(y)$ of a measurement result $y$ is defined by $ur(y) = u(y)/|y|$, where $y$ is not equal to 0.

In statistics, the standard deviation (SD, also represented by the Greek letter sigma $\sigma$ or the Latin letter $s$) is a measure that is used to quantify the amount of variation or dispersion of a set of data values. A low standard deviation indicates that the data points tend to be close to the mean value of the set ($\mu=<x_i>$), while a high standard deviation indicates that the data points are spread out over a wider range of values.

$$\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - \mu)^2}, \text{ where } \mu = \frac{1}{N} \sum_{i=1}^{N} x_i$$
Meaning of uncertainty:

If the probability distribution characterized by the measurement result $y$ and its standard uncertainty $u(y)$ is approximately normal (Gaussian), and $u(y)$ is a reliable estimate of the standard deviation of $y$, then the interval $y - u(y)$ to $y + u(y)$ is expected to encompass approximately 68% of the distribution of values that could reasonably be attributed to the value of the quantity $Y$ of which $y$ is an estimate.

Here $Y$ is the true value (never known exactly) and $y$ is the measured value.

The probability that the true value $Y$ is greater than $y - u(y)$, and is less than $y + u(y)$ is estimated as 68%.

This statement is commonly written as $Y = y \pm u(y)$. 
The interval representing two standard deviations contains 95.4% of all possible true values.
Confidence interval $\mu \pm 3\sigma$ contains 99.7% of possible outcomes.
Use of concise notation:

If, for example, \( v = 1\,234.567\,89 \text{ m/s} \) and \( \Delta v = 0.000\,11 \text{ m/s} \), where m/s is the unit of \( v \), then \( v = (1\,234.567\,89 \pm 0.000\,11) \text{ m/s} \).

A more concise form of this expression, and one that is used sometimes, is \( v = 1\,234.567\,89(11) \text{ m/s} \), where it understood that the number in parentheses is the numerical value of the standard uncertainty referred to the corresponding last digits of the quoted result.

Examples of results which do not make sense (too many digits):

\( v = (1234.5678934534940945 \pm 0.011) \text{ m/s} \)

or \( v = (1234.56 \pm 2) \text{ m/s} \)
If they say $T=63.32456$ F, that would be wrong since they cannot predict temperature with such high precision and the temperature is not stable up to so many significant digits.
Errors in mechanical engineering and industry

Dimensional tolerance

 MMC = Maximum Material Content
 LMC = Least Amount of Material
It is important to know uncertainties in science.

Measurement of the speed of the light:

1675 Ole Roemer: 220,000 km/sec

Does it make sense? What is missing?

Maxwell’s theory prediction:

Speed if light does not depend on the light wavelength; it is universal.

NIST Bolder Colorado $c = 299,792,456.2 \pm 1.1$ m/s.
Use a simple ruler if you do not care about accuracy better than 1mm.

Otherwise you need to use digital calipers.

Probably the natural limit of accuracy can be due to length uncertainty because of temperature expansion. For 53mm $\Delta L \approx 0.012\text{mm}/K$.

Reading Error = $\pm \frac{1}{2}$ (least count or minimum gradation).

How far we have to go in reducing the reading error?
Fluke 8845A multimeter

Example Vdc (reading) = 0.85V
\[ \Delta V = 0.85 \times (1.8 \times 10^{-5}) = 20\, \mu V \]

8846A Accuracy

Accuracy is given as ± (% measurement + % of range)

<table>
<thead>
<tr>
<th>Range</th>
<th>24 Hour (23 ±1 °C)</th>
<th>90 Days (23 ±5 °C)</th>
<th>1 Year (23 ±5 °C)</th>
<th>Temperature Coefficient/ °C Outside 18 to 28 °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 mV</td>
<td>0.0025 + 0.003</td>
<td>0.0025 + 0.0035</td>
<td>0.0037 + 0.0035</td>
<td>0.0005 + 0.0005</td>
</tr>
<tr>
<td>1 V</td>
<td>0.0018 + 0.0006</td>
<td>0.0018 + 0.0007</td>
<td>0.0025 + 0.0007</td>
<td>0.0005 + 0.0001</td>
</tr>
<tr>
<td>10 V</td>
<td>0.0013 + 0.0004</td>
<td>0.0018 + 0.0005</td>
<td>0.0024 + 0.0005</td>
<td>0.0005 + 0.0001</td>
</tr>
<tr>
<td>100 V</td>
<td>0.0018 + 0.0006</td>
<td>0.0027 + 0.0006</td>
<td>0.0038 + 0.0006</td>
<td>0.0005 + 0.0001</td>
</tr>
<tr>
<td>1000 V</td>
<td>0.0018 + 0.0006</td>
<td>0.0031 + 0.001</td>
<td>0.0041 + 0.001</td>
<td>0.0005 + 0.0001</td>
</tr>
</tbody>
</table>
The accuracy of an experiment is a measure of how close the result of the experiment comes to the true value.

Precision refers to how closely individual measurements agree with each other.
• **Systematic Error:** reproducible inaccuracy introduced by faulty equipment, calibration, technique, model, drifts, 1/f noise.

• **Random errors:** Indefiniteness of results due to finite precision of experiment. Errors can be reduced by repeating the measurement and averaging.

Sources of systematic errors: poor calibration of the equipment, changes of environmental conditions, imperfect method of observation, drift and some offset in readings etc.

Example #1: measuring of the DC voltage

\[ U = RI + E_{\text{off}} \]

\[ U = R \times I \]
Example #3: poor calibration

Measuring of the speed of the second sound in superfluid He4

Published data

\[ T_\lambda = 2.17\text{K} \]

P403 results

\[ T_\lambda = 2.1\text{K} \]
A statistical process is described through a Poisson Distribution if:

- **Random process** → for a given nucleus, probability for a decay to occur is the same in each time interval.
- **Universal probability** → the probability to decay in a given time interval is same for all nuclei.
- **No correlation between two instances** (the decay of one nucleus does not change the probability for a second nucleus to decay.)

The Poisson distribution is given by:

$$ P_n(t) = \frac{(rt)^n}{n!} e^{-rt} \quad n = 0, 1, 2, ... $$

- $r$: decay rate [counts/s]  
- $t$: time interval [s]  
- $P_n(rt)$: Probability to have $n$ decays in time interval $t$
$P_n(t) = \frac{(rt)^n}{n!} e^{-rt} \quad n = 0,1,2,...$

$r$: decay rate [counts/s]  \hspace{1cm} t$: time interval [s]  \hspace{1cm} \Rightarrow P_n(rt)$: Probability to have $n$ decays in time interval $t$

**Properties of the Poisson distribution:**

$$\sum_{n=0}^{\infty} P_n(rt) = 1 \quad \text{, probabilities sum to 1}$$

$$<n> = \sum_{n=0}^{\infty} n \cdot P_n(rt) = rt \quad \text{, the mean}$$

$$\sigma = \sqrt{\sum_{n=0}^{\infty} (n - <n>)^2 P_n(rt)} = \sqrt{rt} \quad \text{, standard deviation}$$
$P_n(t) = \frac{(rt)^n}{n!} e^{-rt} \quad n = 0, 1, 2, \ldots$

Poisson and Gaussian distributions

Gaussian distribution:

$$P_n(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}$$

Carl Friedrich Gauss
(1777–1855)
Normal (Gaussian) distribution

\[ P_n(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{\frac{(x-\bar{x})^2}{2\sigma^2}} \]
Measurement in presence of noise

Source of noisy signal

Actual measured values:
- 4.89855
- 5.25111
- 2.93382
- 4.31753
- 4.67903
- 3.52626
- 4.12001
- 2.93411

Expected value 5V
Measurement in presence of noise

Error in the mean is given as $$\frac{\sigma_0}{\sqrt{N}}$$
\[ U = x_c \pm \frac{\sigma}{\sqrt{N}} \]

- Standard deviation (\( \sigma \))
- Number of samples (\( N \))

**Result**

For \( N = 10^6 \) \( U = 4.999 \pm 0.001 \)  
0.02\% accuracy
The standard error equals the standard deviation divided by the square root of the sample size (=number of measurements). In other words, the standard error of the mean is a measure of the dispersion of sample means around the population mean.

\[
U = x_c \pm \frac{\sigma}{\sqrt{N}}
\]

\(\sigma\) - standard deviation

N – number of samples
Ag $\beta$ decay

$^{108}$Ag $t_{1/2} = 157$ s
$^{110}$Ag $t_{1/2} = 24.6$ s

Model ExpDec2
Equation $y = A_1 \exp(-x/t_1) + A_2 \exp(-x/t_2) + y_0$
Reduced Chi-Sqr 1.43698
Adj. R-Square 0.96716

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_0$</td>
<td>0.02351</td>
<td>0.95435</td>
</tr>
<tr>
<td>$A_1$</td>
<td>104.87306</td>
<td>12.77612</td>
</tr>
<tr>
<td>$t_1$</td>
<td>177.75903</td>
<td>18.44979</td>
</tr>
<tr>
<td>$A_2$</td>
<td>710.01478</td>
<td>25.44606</td>
</tr>
<tr>
<td>$t_2$</td>
<td>30.32479</td>
<td>1.6525</td>
</tr>
</tbody>
</table>

$y = A_1 \exp\left(-\frac{t}{t_1}\right) + A_2 \exp\left(-\frac{t}{t_2}\right) + y_0$
Fitting. Analysis of the residuals

Ag β decay

Test 1. Fourier analysis

No pronounced frequencies found
Fitting. Analysis of the residuals

Ag β decay

Test 1. Autocorrelation function

Correlation function

\[ y(m) = \sum_{n=0}^{\infty} f(n) g(n - m) \]

autocorrelation function

\[ y(m) = \sum_{n=0}^{M-1} f(n) f(n - m) \]
Fitting. Analysis of the residuals. Non “ideal” case

Ag β decay

<table>
<thead>
<tr>
<th>Model</th>
<th>ExpDec2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equation</td>
<td>[ y = A_1 e^{(-t/t_1)} + A_2 e^{(-t/t_2)} + y_0 ]</td>
</tr>
</tbody>
</table>

| Reduced Chi-Sqr | 100.10041 |
| Adjusted R-Square | 0.99181 |

<table>
<thead>
<tr>
<th>F</th>
<th>Value</th>
<th>Standard Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_0 )</td>
<td>5.18284</td>
<td>1.96542</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>130.85655</td>
<td>20.27379</td>
</tr>
<tr>
<td>( t_1 )</td>
<td>145.89449</td>
<td>21.82649</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>792.62197</td>
<td>19.21953</td>
</tr>
<tr>
<td>( t_2 )</td>
<td>27.83939</td>
<td>1.30697</td>
</tr>
</tbody>
</table>

### Clear experiment

| \( t_1(s) \) | 177.76 |
| \( t_2(s) \) | 30.32 |

### Data + “noise”

| \( t_1(s) \) | 145.89 |
| \( t_2(s) \) | 27.94 |
Fitting. Analysis of the residuals. Non “ideal” case

Ag β decay

Histogram does not follow the normal distribution and there is frequency of 0.333 is present in spectrum
Conclusion: fitting function should be modified by adding an additional term:

\[ y(t) = y_0 + A_1 \exp\left(-\frac{t}{t_1}\right) + A_2 \exp\left(-\frac{t}{t_2}\right) + A_3 \sin(\omega t + \theta) \]
Fitting. Analysis of the residuals. Non “ideal” case

<table>
<thead>
<tr>
<th></th>
<th>Clear experiment</th>
<th>Data + noise</th>
<th>Modified fitting</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_1$(s)</td>
<td>177.76</td>
<td>145.89</td>
<td>172.79</td>
</tr>
<tr>
<td>$t_2$(s)</td>
<td>30.32</td>
<td>27.94</td>
<td>30.17</td>
</tr>
</tbody>
</table>
Error propagation

\[ y = f(x_1, x_2, \ldots, x_n) \]

\[ \Delta f(x_i, \Delta x_i) = \sqrt{\sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \right)^2 \cdot \Delta x_i^2} \]
Derive resonance frequency $f$ from measured inductance $L \pm \Delta L$ and capacitance $C \pm \Delta C$

$$f(L, C) = \frac{1}{2\pi} \sqrt{\frac{1}{LC}}$$

$L_1 = 10 \pm 1\text{mH}, \quad C_1 = 10 \pm 2\text{μF}$

$$\Delta f(L, C, \Delta L, \Delta C) = \sqrt{\left[\frac{\partial f}{\partial L}\right]^2 \cdot \Delta L^2 + \left[\frac{\partial f}{\partial C}\right]^2 \Delta C^2}$$

$$\frac{\partial f}{\partial L} = \frac{-1}{4\pi} C^{-\frac{1}{2}} L^{-\frac{3}{2}};$$

$$\frac{\partial f}{\partial C} = \frac{-1}{4\pi} L^{-\frac{1}{2}} C^{-\frac{3}{2}}$$

Results:

$f(L_1, C_1) = 503.29212104487\text{Hz}$

$\Delta f = 56.26977\text{Hz}$

$f(L_1, C_1) = 503.3 \pm 56.3\text{Hz}$
Figure 3. Magnetization (M/Ms) of Mn3 single crystal versus applied magnetic field with the sweeping rate of 0.003 T/s at different temperatures. The inset shows ZFC and FC curves.

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Figure 2. Normalized conductivity vs temperature for three 250-nm-thick K0.33WO3–y films on YSZ substrates. The films are annealed in vacuum at different temperatures, with properties shown in the inset table. The units of T\text{anneal} are degrees Celcius, \(\sigma_0\) is given in 1/mΩcm, \(n\) in /cm\(^3\), and Tc in degrees Kelvin.

Phys. Rev. B 89, 184501
Figure 1. Normalized residuals of the combined dE/dx for antideuteron candidates in the Onpeak \( \Upsilon(2S) \) data sample, with fit PDFs superimposed. Entries have been weighted, as detailed in the text. The solid (blue) line is the total fit, the dashed (blue) line is the d\(^-\) signal peak, and the dotted (red) line is the background.

Phys. Rev. D 89, 111102(R)
Figure 10(ii): lambda versus T for indium film with thickness 300 nm. Input voltage is 0.2v. Critical temperature (b) and penetration depth (A) at temperature 0 K is determined.

Spring 2014.
Figure 8: Coincidence Rate vs. Detector Angle for 22Na correlation measurement.

Figure 11: Temperature dependence of energy gap in Sn. Red line is BCS theory.

Summer 2019.