Driven Torsional Oscillator

Physics 401, Spring 2016
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Agenda

1. Driven torsional oscillator. Equations
2. Setup. Kinematics
3. Resonance
4. Beats
5. Nonlinear effects
6. Comments
Before starting the torsional oscillator discussion let we take a look on some historical examples showing how dangerous the resonance in mechanical systems can be
Torsional oscillations. Resonance.

Tacoma (WA) Narrows Bridge Disaster
Tacoma (WA) Narrows Bridge, 1940
Torsional oscillations. Resonance.

Tacoma (WA) Narrows Bridge, 1940
Tacoma (WA) Narrows Bridge, 1940
Mechanical Resonance.

Egyptian Bridge disaster. 20 January 1905, St. Petersburg, Russia.
Mechanical Resonance.

Egyptian Bridge disaster. 20 January 1905, St. Petersburg, Russia.
Mechanical Resonance.

Egyptian Bridge disaster. 20 January 1905, St. Petersburg, Russia.
“Dancing Bridge” in Volgograd (Russia) (record from 2\textsuperscript{st} May 2010. 4.4 miles long).
In autumn 2011, 12 semi-active tuned mass dampers were installed in the bridge. Each one consists of a mass 5,200 kg (11,500 lb), a set of compression springs and a magnethoreological damper.
The goals: (i) analyze the response of the damped driven harmonic oscillator to a sinusoidal drive. (ii) transient response and (iii) steady-state solution.

Angular displacement: \( \theta_0 \cos(\omega t) \);
Torque: \( K\lambda \theta_0 \cos(\omega t) \)

\[ \lambda = \frac{L_1}{L_1 + L_2} \]

Viscous damping
Torque by motor
Driven torsional oscillator

Motor

Pendulum
\[ I \ddot{\theta} + K \theta + R \dot{\theta} = \tau_m = K\lambda \theta_0 \cos(\omega t) \]

Solutions: sum of (1) Transient solution + (2) steady solution due to torque \( \tau_m \)

(1) Transient solution (1\textsuperscript{st} week experiment)

\[ I \ddot{\theta} + R \dot{\theta} + K \theta = 0 \]
\[ \theta(t) = Ae^{-at} \cos(\omega_1 t - \phi) \]
\[ a = \frac{R}{2I} \]
\[ \omega_0 = \sqrt{\frac{K}{I}} \]
\[ \omega_1 = \sqrt{\omega_0^2 - a^2} \]

The homogeneous equation of motion
Transient solution
Attenuation constant
Natural (angular) frequency
Damped (angular) frequency
Steady-state solution

\[ \theta_i(t) = |A|e^{-at} \cos(\omega_1 t + \phi) \rightarrow \omega_1 = \sqrt{\omega_0^2 - a^2} \]

Transient solution

Once this response dies away in time the system response only on the frequency of drive \( \omega \)

Initially the system responds on the characteristic frequency \( \omega_1 \)

So the steady-state solution must have the similar time dependence as the drive

\[ \theta_{ss}(t) = \text{Re}\left( \theta(\omega)e^{i\omega t} \right) \]

Substituting \( \theta_{ss}(t) \) in equation of motion we will find the equations for \( \theta(\omega) \)

\[ \theta(\omega) = \frac{\lambda \omega_0^2 \theta_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2 a^2}} e^{-i\beta(\omega)} \]

and

\[ \beta(\omega) = \tan^{-1}\left( \frac{2\omega a}{\omega_0^2 - \omega^2} \right) \]
Steady-state solution. Summary.

\[ I\ddot{\theta} + K\theta + R\dot{\theta} = \tau_m = K\lambda \theta_0 \cos(\omega t) \]

(2) steady solution

\[ \theta_s(t) = B(\omega) \cos(\omega t - \beta(\omega)) \]

Steady state solution

\[ B(\omega) = \frac{\lambda \theta_0 \omega^2}{\sqrt{(\omega_0^2 - \omega^2)^2 + \omega^2 \gamma^2}} \]

Amplitude function

\[ \tan \beta(\omega) = \frac{\omega \gamma}{\omega_0^2 - \omega^2} \]

Phase function

\[ \gamma = \frac{R}{I} = 2 \frac{R}{2I} = 2a \]

Damping constant
General solution

time domain form for steady-state solution will be

\[ \theta_{ss}(t) = \frac{\lambda \omega_0^2 \theta_0}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2 a^2}} \cos(\omega t - \beta(\omega)) \]

General solution for equation of motion consist of the sum of sum of two components:

\[ \theta(t) = \theta_t(t) + \theta_{ss}(t) \]

\[ \theta(t) = \theta_t(t) + \theta_{ss}(t) = A e^{-at} \cos(\omega_1 t - \phi) + B \cos(\omega t - \beta(\omega)) \]

Coefficients \( A \) and \( \phi \) could be determined from initial conditions.
Fitting function:

\[ \theta(f) = \frac{A \cdot f_0^2}{\sqrt{(f_0^2 - f^2)^2 + \gamma^2 f^2}} \]

\[ \omega = 2\pi f; \quad \gamma = 2a \]

To create a new fitting function go "Tools" → "Fitting Function Builder" or press F8
Scanning the driving frequency we can measure the amplitude of the pendulum oscillating and the phase shift.

Both parameters Amplitude and phase can be defined by DAQ program or using Origin.
Resonance. Amplitude of the Angular Displacement.

Amplitude

\[ |\theta_{ss}(t)| = \frac{\lambda \omega_0^2 \theta_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2a^2}} \]

At resonance \( \omega = \omega_0 \)

\[ |\theta_{ss}(t)| = \frac{\lambda \omega_0 \theta_0}{2a} = \lambda \theta_0 \cdot Q \]

Combination of high initial amplitude \( \theta_0 \), and high quality \( Q \) or low damping factor \( a \) could be result of the destruction of the mechanical system
For correct representation of the resonance curve take care about choosing of the step size in frequency.

\[ f_0 = 0.495 \text{Hz} \]
There are two parameters used to measure the rate at which the oscillations of a system are damped out. One parameter is the logarithmic decrement \( \delta \), and the other is the quality factor, \( Q \).

\[ \delta = \ln \left( \frac{\theta(t_{\text{max}})}{\theta(t_{\text{max}} + T_1)} \right) = \ln \left( \frac{e^{-at_{\text{max}}}}{e^{-a(t_{\text{max}} + T_1)}} \right) = aT_1. \]

\( \theta(t_{\text{max}}) \) and \( \theta(t_{\text{max}} + T_1) \) are points on the graph.

\[ \delta = \ln \left( \frac{8.49}{7.35} \right) \approx 0.144 \]

\[ Q = 2\pi \frac{\text{total stored energy}}{\text{decrease in energy per period}} \]

\[ Q = \frac{\omega_1}{R/I} = \frac{\omega_1}{2a} = \frac{\pi \omega_1}{2a} = \frac{\pi}{a} \frac{1}{T_1} = \frac{\pi}{\delta} \]

\( Q \approx 21.8 \)
It can be shown that $Q$ can be calculated as $\omega_1/\Delta\omega$ or $f_1/\Delta f$. $\Delta\omega$ is bandwidth of the resonance curve on the half power level or $\theta_{\max}/\sqrt{2}$ for amplitude graph.

Here $Q \approx 7.9$
Beats. Theory.

Consider sum of two harmonic signals of frequencies $\omega_1$ and $\omega_2$

\[ y_1 = A \sin(\omega_1 t + \varphi_1); \quad y_2 = B \sin(\omega_2 t + \varphi_2) \]

In case $A = B$ \[ y = y_1 + y_2 = 2A \sin \left( \frac{\omega_1 + \omega_2}{2} t + \beta_1 \right) \cos \left( \frac{\omega_1 - \omega_2}{2} t + \beta_2 \right) \];

$\beta_1 = \frac{\varphi_1 + \varphi_2}{2}; \quad \beta_2 = \frac{\varphi_1 - \varphi_2}{2}$

If $\omega_1 \approx \omega_2 \approx \frac{\omega_1 + \omega_2}{2} = \omega$ \quad and \quad $\frac{\omega_1 - \omega_2}{2} = \Omega$

\[ y = 2A \cos(\Omega t + \beta_2) \sin(\omega t + \beta_1) \]
More general case $A \neq B$ \( \omega_1 \) and \( \omega_2 \)

\[ y_1 = A \sin(\omega_1 t); \quad y_2 = B \sin((\omega_1 + \alpha) t) \]

\[ y = y_1 + y_2 = C \sin((\omega + \beta) t) \quad \text{where} \quad C = \sqrt{A^2 + B^2 + 2AB \cos(\alpha t)} \]

\[ \beta = \tan^{-1} \left( \frac{B \sin(\alpha t)}{A + B \cos(\alpha t)} \right) + \begin{cases} 0 & \text{if } A + B \cos(\alpha t) \geq 0 \\ \pi & \text{if } A + B \cos(\alpha t) < 0 \end{cases} \]
Two peaks corresponding $\omega$ and $\omega_1$

Use Origin to analyze the frequency spectrum!
Beats dying in time. How fast – it depends on damping. When you will work on resonance data – wait until you will see the steady state oscillations.

$$\theta(t) = \theta_t(t) + \theta_{ss}(t) = Ae^{-at} \cos(\omega_1 t - \phi) + B \cos(\omega t - \beta(\omega))$$

$$\theta_t(t) \to 0$$
\[ \theta(t) = \theta_{t}(t) + \theta_{ss}(t) = Ae^{-at} \cos(\omega_{1}t - \phi) + B \cos(\omega t - \beta(\omega)) \]

This can be seen well from "envelope" plot.

\[ \theta_{t}(t) \rightarrow 0 \]

Origin 8.6: Analysis → Signal Processing → Envelope
\[ \theta(t) = \theta_t(t) + \theta_{ss}(t) = Ae^{-at} \cos(\omega_1 t - \phi) + B \cos(\omega t - \beta(\omega)) + C \]

First let we apply FFT to find \( \omega_1 \) and \( \omega \)

Result: \( \omega_1 = 3.1402 \text{ rad}^{-1} \) and \( \omega = 2.8298 \text{ rad}^{-1} \)
\[ \theta(t) = \theta_t(t) + \theta_{ss}(t) = A e^{-\frac{t}{t_0}} \cos(\omega_1 t - \phi) + B \cos(\omega t - \beta(\omega)) + C \]

8 fitting parameters

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<tr>
<th>Value</th>
<th>Standard Err</th>
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<td>C</td>
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From fitting:

\[ \omega_1 = 3.1402 \text{ rad}^{-1} \] and \[ \omega = 2.8298 \text{ rad}^{-1} \]

Result from FFT:

Possible origin of “extra” peaks:
(i) Nonlinear behavior of pendulum
(ii) Not a single frequency driving force provided by motor
(iii) Not ideal fitting function
$$\theta(t) = \theta_i(t) + \theta_{ss}(t) = Ae^{-at} \cos(\omega_1 t - \phi) + B \cos(\omega t - \beta(\omega))$$

$$\theta_i(t) \rightarrow 0$$

We also can analyze the decrease of the amplitude of the $\omega_1$ component by analyzing the spectrum as a function of time.
Beats. RLC Experiment.
Beats. RLC Experiment.

Find peaks

Envelope

FFT

310kHz
340kHz
Beats. Experiment. More complicated case.

In the case of driving frequency $f_d = f1/N$ where $N$ is integer we can observe more complicated motion of the pendulum.

\[ \omega_d \sim \omega_0 / 3 \]

\[ f_0 = 0.4891 \text{ Hz} \]
\[ f_d = 0.163 \text{ Hz} \]

\[ f_0 - f_d \quad f_0 \quad f_0 + f_d \]
In the case of driving frequency \( f_d = f_1/N \) where \( N \) is integer we can observe more complicated motion of the pendulum.
Detailed analyzes* shows that even if $\phi = \phi_0 \sin(\omega t)$ the driving torque contains several harmonics of $\omega$

*P. Debevec (UIUC, Department of Physics)