Frequency Domain Analysis of Linear Circuits Using Synchronous Detection

Physics 401, Fall 2017.
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in 1822, Jean Baptiste Fourier developed the theory that shows that any real waveform can be represented by the sum of sinusoidal waves.

Let we try to create the square wave as a sum of sine waves of different frequencies

Jean Baptiste Joseph Fourier
(1768 – 1830)
\begin{align*}
A_1 \sin(2\pi \omega t) & \quad \quad A_1 \sin(2\pi \omega t) + A_3 \sin(2\pi 3\omega t + \phi_3) \\
& + A_5 \sin(2\pi 5\omega t + \phi_5) & \quad \quad A_1 \sin(2\pi \omega t) + A_3 \sin(2\pi 3\omega t + \phi_3) + \\
& + A_5 \sin(2\pi 5\omega t + \phi_5) + A_7 \sin(2\pi 7\omega t + \phi_7) \\
\end{align*}
The continues Fourier transformation of the signal \( h(t) \) can be written as:

\[
H(f) = \int_{-\infty}^{+\infty} h(t)e^{2\pi jft} dt; \quad j = \sqrt{-1}
\]

\( H(f) \) represents in frequency domain mode the time domain signal \( h(t) \).

Equation for inverse Fourier transform gives the correspondence of the infinite continues frequency spectra to the corresponding time domain signal.

\[
h(t) = \int_{-\infty}^{+\infty} H(f)e^{-2\pi jft} df
\]

In real life we working with discrete representation of the time domain signal recorded during a finite time.
It comes out that in practice more useful is the representation the frequency domain pattern of the time domain signal $h_k$ as sum of the frequency harmonic calculated as:

$$H_n = H(f_n) = \frac{1}{N} \sum_{k=0}^{N-1} h_k e^{2\pi kn/N}$$

$\Delta$ is the sampling interval, $N$ – number of collected points
Discrete Fourier Transform

For periodic signals with period $T_0$:

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left( \frac{2\pi nt}{T_0} \right) + \sum_{n=1}^{\infty} b_n \sin \left( \frac{2\pi nt}{T_0} \right)$$

$$a_n = \frac{2}{T_0} \int_0^{T_0} F(t) \cos \left( \frac{2\pi nt}{T_0} \right) dt; \quad b_n = \frac{2}{T_0} \int_0^{T_0} F(t) \sin \left( \frac{2\pi nt}{T_0} \right) dt;$$

$$a_0 = \frac{2}{T_0} \int_0^{T_0} F(t) dt;$$
Now how I found the amplitudes of the harmonics to compose the square wave signal from sine waves of different frequencies.

**Time domain signal**

Decomposition the signal into the sine wave harmonics. The only modulus's of the harmonics amplitudes are presented in this picture.
We are applying the sine wave signal to the tested object and measuring the response. Varying the frequency, we can study the frequency properties of the system.
Now about the most powerful tool which can be used in frequency domain technique.

**Signal in**  \( \rightarrow \) **Signal amplifier**  \( \rightarrow \) **Signal monitor**

**Reference in**  \( \rightarrow \) **VCO**  \( \rightarrow \) **PSD**

**Reference out**  \( \rightarrow \) **Low-pass filter**  \( \rightarrow \) **DC amplifier**  \( \rightarrow \) **output**

*PSD - phase sensitive detector; **VCO - voltage controlled oscillator*

John H. Scofield
*American Journal of Physics 62 (2)*
129-133 (Feb. 1994).
The DC output signal is a magnitude of the product of the input and reference signals. AC components of output signal are filtered out by the low-pass filter with time constant $\tau$ (her $\tau=RC$).
Lock-in Amplifier. What is the Advantage of Using Synchronous Detecting?

DMM, lock-in etc.

AC Results as DC voltage corresponding $U_{\text{AMP}}$, $U_{\text{RMS}}$ ...

$U(t)$

Measuring equipment

$U(t)$

$R$

$C$

$1$

$2$

$9/11/2017$
Clean sine wave – no “noise”

We need to measure the amplitude/rms value of the sine wave

\[ U_{DC} = 0.63643 \text{V} \]
Lock-in Amplifier. What is the Advantage of Using Synchronous Detecting?

“Noisy” sine wave

$U_{DC} = 0.64208\text{V}$

compare to

$U_{DC} = 0.63643\text{V}$
Lock-in Amplifier. What is the Advantage of Using Synchronous Detecting?

2 Clear sine wave – no “noise”

U_{DC} = 0.63643V

“Noisy” sine wave

U_{DC} = 0.63643V
Lock-in Amplifier. Phase shift.

\[ V_0 \sin(\omega t + \phi) \]

\[ \phi = \pi/4, \quad V_{out} = 0.72 V_{in} \]

\[ V_{in} = \sin(\omega t + \pi/4) \]
In many scientific applications it is a great advantage to measure both components \((E_x, E_y)\) of the input signal. We can use two lock-ins to do this or we can measure these values in two steps providing the phase shift of reference signal \(0\) and \(\pi/2\). Much better solution is to use the lock-in amplifier equipped by two demodulators.

\[
E_{\text{in}} = E_0 \sin(\omega t + \phi)
\]

\[
E_x = E_0 \sin(\omega t) \cos(\omega t)
\]

\[
E_y = E_0 \cos(\omega t)
\]
Digital Lock-in Amplifier

Input amplifier

ADC

DSP

DAC

Digital interface

External reference signal

Analog outputs

Internal Function generator

$e_{in}$

$A \sin(\omega t + \phi)$
SR830. Digital Lock-in Amplifier

In SR830 manual you can find the chapter dedicated to general description of the lock-in amplifier idea.

WHAT IS A LOCK-IN AMPLIFIER?

Lock-in amplifiers are used to detect and measure very small AC signals - all the way down to a few nanovolts! Accurate measurements may be made even when the small signal is obscured by noise experiment at the reference frequency. In the diagram below, the reference signal is a square wave at frequency $\omega_R$. This might be the sync output from a function generator. If the sine output from

\[\text{engr-file-03\PHYINST\APL Courses\PHYCS401\Common\EquipmentManuals}\]
Digital Lock-in amplifier. SR830

Time constant
And output filter sensitivity
Channel#1
Channel#2
Auto functions

Inputs
Notch filter settings
Analog outputs
Function generator

9/11/2017
illinois.edu
Experiments. Main idea. Investigating the frequency response of circuit.

\[ \tilde{V}_{\text{in}}(\omega) \rightarrow H(\omega) \rightarrow \tilde{V}_{\text{out}}(\omega) \]

Frequency domain representation of the system

Response function \[ \tilde{H}(\omega) = \frac{\tilde{V}_{\text{out}}(\omega)}{\tilde{V}_{\text{in}}(\omega)} \]

Linear systems are those that can be modeled by linear differential equations.
Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit

Setup for measurement of the transfer function of the RLC circuit.

\[ h(\omega) = \frac{e_{\text{out}}}{e_{\text{in}}} \]

Example 1. High-pass filter.

\[ V_{out}(\omega) = H(\omega) \ast V_{in}(\omega) = V_{in}(\omega) \frac{\tilde{Z}_2(\omega)}{\tilde{Z}_1(\omega) + \tilde{Z}_2(\omega)} \]

Applying the Kirchhoff Law to this simple network.
Experiments. Calculation of the Response Function in Frequency Domain Mode. High-pass Filter

Ideal case

\[ \tilde{Z}_R = R \]
\[ \tilde{Z}_L = j\omega L \]
\[ \tilde{Z}_C = \frac{1}{j\omega C} = -\frac{j}{\omega C} \]

More realistic

\[ \tilde{Z}_R = R + \ldots \]
\[ \tilde{Z}_L = j\omega L + R_L \]
\[ \tilde{Z}_C = \frac{1}{j\omega C} = \frac{1}{j\omega C + R_C^{-1}} \]

\[ \tilde{V}_{out}(\omega) = \tilde{H}(\omega) \ast \tilde{V}_{in}(\omega) = \tilde{V}_{in}(\omega) \frac{\tilde{Z}_2(\omega)}{\tilde{Z}_1(\omega) + \tilde{Z}_2(\omega)} \]
Experiments. Calculation of the Response Function in Frequency Domain Mode. High-pass Filter

\[ \tilde{H}(\omega) = H_R(\omega) + jH_I(\omega) = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC} = \frac{j\omega \tau}{1 + j\omega \tau} = \frac{\omega \tau}{1 + \omega^2 \tau^2} (\omega \tau + j); \]

where \( \tau = RC = \omega_c^{-1}; \)

\[ |\tilde{H}(\omega)| = \sqrt{H_R^2 + H_I^2} = \frac{\omega \tau}{\sqrt{1 + (\omega \tau)^2}}; \quad \theta(\omega) = \arctan \left( \frac{H_I(\omega)}{H_R(\omega)} \right) = \arctan \left( \frac{1}{\omega \tau} \right) \]

\( \tau \) – time constant of the filter
\( \omega_c \) - cutoff frequency
Experiments. Calculation of the Response Function in Frequency Domain Mode. High-pass Filter

\[ V_{\text{out}}(\omega) \]

\[ V_{\text{in}}(\omega) \]

Hi-pass filter: \( C=20\text{nF}, R=20\text{K} \)

\[ f_c \approx 398\text{Hz} \]

\[ V_0 \]

\[ \frac{V_0}{\sqrt{2}} \]

\[ |\tilde{H}(\omega)| = \sqrt{H_R^2 + H_I^2} = \frac{\omega\tau}{\sqrt{1+(\omega\tau)^2}} \]

\[ f_c = \frac{\omega_c}{2\pi} = \frac{1}{2\pi\tau} = \frac{1}{2\pi RC} \]
Fitting parameters: $V_0$, $\tau$, $V_{off}$

Fitting function

$$\tilde{V}_{out} = \tilde{V}_{in} \ast \tilde{H}(\omega) = V_0 \ast \frac{\omega \tau}{\sqrt{1 + (\omega \tau)^2}}; \quad \tau = RC$$

\[ \theta(\omega_c) = \arctan(1) = 45^\circ \]

\[ f_c \approx 398\text{Hz} \]
Experiments. Calculation of the Response Function in Frequency Domain Mode. Low-pass Filter

\[ \tilde{H}(\omega) = H_R(\omega) + jH_I(\omega) = \frac{1}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = \frac{1}{1 + j\omega\tau} = \frac{1 - j\omega\tau}{(1 + \omega^2\tau^2)}; \]

where \( \tau = RC = \omega_c^{-1}; \)

\[ |\tilde{H}(\omega)| = \sqrt{H_R^2 + H_I^2} = \frac{1}{\sqrt{1 + (\omega\tau)^2}}; \quad \theta(\omega) = \arctan \left( \frac{H_I(\omega)}{H_R(\omega)} \right) = -\arctan(\omega\tau) \]
Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit.

\[ U_C = U_{in} \cdot \frac{Z_C}{Z_C + Z_L + R} = \]
\[ = \frac{1}{j\omega C + j\omega L + R} \]
Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit.

\[
H = \frac{U_C}{U_{in}} = \frac{1}{(1 - \omega^2 LC) + j\omega CR} = \frac{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right) - j\omega CR}{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \omega^2 C^2 R^2} \times;
\]

\[
\omega_0 = \frac{1}{\sqrt{LC}}; \nu \equiv \frac{\omega}{\omega_0}; Q = \frac{1}{R \sqrt{C}} ;
\]

\[
H = \frac{(1 - \nu^2) - j\frac{\nu}{Q}}{(1 - \nu^2)^2 + \frac{\nu^2}{Q^2}} ; \theta = -\tan^{-1}\left(\frac{\nu}{Q\left(1 - \nu^2\right)}\right)
\]
The resonance curves obtained on RLC circuits with different damping resistors.
Application of the Lock-in Amplifier for Study of the Transfer Function of the RLC Circuit

The resonance curves obtained on RLC circuits with different damping resistors

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Fitting. RLC Resonance Circuit.

\[ |H| = \sqrt{(1 - \gamma^2)^2 + \left(\frac{\gamma}{Q}\right)^2} \]

; \gamma = \frac{\omega}{\omega_0}

fitting function for $|H|

variable parameters: $\omega_0$ and $Q$
Actual damping resistance is a sum of $R$, $R_L$ (resistance of the coil) and $R_{out}$ (output resistance of the function generator).

Actual $R$ calculated from fitting pars is $\approx 88.8 \Omega$, what is reasonable close to $85.8 \Omega$. 

$R = 0$; $R_L = 35.8 \Omega$; $R_{out} = 50 \Omega$
Fitting. RLC Resonance Circuit.

\[ \theta = \tan^{-1}\left( \frac{Y}{X} \right) \]

measured

\[ \theta = -\tan^{-1}\left( \frac{\gamma}{Q\left(1 - \gamma^2\right)} \right); \gamma = \frac{\omega}{\omega_0} \]

fitting function

variable parameters: \( \omega_0 \) and \( Q \)
From Time Domain to Frequency Domain. Experiment.

Wavetek

Lock-in SR830

Out

Sync

input

Reference in

F(t) – periodic function \( F(t) = F(t + T_0) \):

\[
V = V_0 \begin{cases} 
0 < t \leq \frac{T_0}{2} \\
- V_0 \left( \frac{T_0}{2} < t \leq T_0 \right)
\end{cases}
\]

\[
a_n = \frac{2}{T_0} \int_0^{T_0} F(t) \cos \left( \frac{2\pi nt}{T_0} \right) dt;
\]

\[
b_n = \frac{2}{T_0} \int_0^{T_0} F(t) \sin \left( \frac{2\pi nt}{T_0} \right) dt;
\]

\[
a_0 = \frac{2}{T_0} \int_0^{T_0} F(t) dt
\]

Time domain pattern

Frequency domain?
From Time Domain to Frequency Domain. Experiment with SR830. Results.

Time domain

Spectrum measured by SR 830 lock-in amplifier

Frequency domain
From Time Domain To Frequency Domain. FFT using Origin. Results.

Data file can be used to convert time domain to frequency domain

Time domain taken by Tektronix scope
From Time Domain to Frequency Domain. FFT using Origin. Results.

Time domain taken by Tektronix scope

Spectrum calculated by Origin. Accuracy is limited because of the limited resolution of the scope
From Time Domain to Frequency Domain. Using of the Math Option of the Scope.

Time domain taken by Tektronix scope

Spectrum calculated by Tektronix scope. Accuracy is limited because of the limited resolution of the scope.
From Time Domain to Frequency Domain. Using of the Math Option of the Scope.

Spectrum of the square wave signal

Spectrum of the pulse signal
Origin templates for the this week Lab:

- Graph 1: Plot of \( R \) (V) vs. \( f \) (Hz) with a range from \(-160\) to \(-10^{1}\) and \(-10^{1}\) to \(-10^{2}\) for the vertical and horizontal axes, respectively.

- Graph 2: Plot of Signal (V) vs. time (ms) with a range from 0 to 4 for the time axis.
References:


- You can find a soft copy of this book in:
  - \engr-file-03\phyinst\APL Courses\PHYCS401\Experiments