Coupled Linear Oscillators

$$\mathbf{M}\ddot{\vec{q}} = -\mathbf{K}\vec{q} \qquad T = \frac{1}{2}\mathbf{M}_{ij}\dot{q}_i\dot{q}_j \qquad U = \frac{1}{2}\mathbf{K}_{ij}q_i q_j \qquad \mathbf{M}_{ij} = \left.\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j}\right|_{\vec{q}=0} \qquad \mathbf{K}_{ij} = \left.\frac{\partial^2 U}{\partial q_i \partial q_j}\right|_{\vec{q}=0}$$

Transverse oscillations of taut string: $k_T = \left.\frac{\text{tension}}{\text{length}}\right|_{\text{equilib}}$

Inner Product Space description and **Normal Coordinates** $\qquad \vec{q}$ = column vector of generalized coord

- **Space** : $|\vec{q}(t)\rangle \equiv$ solutions of a linear oscillator system
- **Inner Product** : $\langle \vec{q}_1 | \vec{q}_2 \rangle \equiv \vec{q}_1^{\,T}\mathbf{M}\vec{q}_2$ and associated **magnitude** : $|\vec{q}|^2 \equiv \langle \vec{q} | \vec{q} \rangle$
- **Basis** : $|\hat{a}_m\rangle$ of eigenvectors defined by $\mathbf{K}\vec{a}_m = \omega_m^2 \mathbf{M}\vec{a}_m$ and normalization $\hat{a}_m \equiv \vec{a}_m / |\vec{a}_m|$
- **Basis is Orthonormal** : $\langle \hat{a}_n | \hat{a}_m \rangle = \delta_{nm}$
- **Completeness** for $\vec{q}(t)$ and **Normal Coordinates** ξ_m :

  ξ_m is the **component** of $\vec{q}$ along mode m : $\quad |\vec{q}(t)\rangle = \sum_{\text{modes } m} |\hat{a}_m\rangle\langle \hat{a}_m | \vec{q}(t)\rangle \equiv \sum_{\text{modes } m} \hat{a}_m \xi_m(t)$

  ξ_m is **projected out** of $\vec{q}$ by : $\quad \xi_m(t) = \langle \hat{a}_m | \vec{q}(t) \rangle = \tilde{A}_m e^{i\omega_m t} = A_m \cos(\omega_m t - \delta_m)$

- **Transformation** betw q-space and ξ-space :

  vectors : $\quad \vec{\xi} = \mathbf{R}\vec{q} \qquad \vec{q} = \mathbf{R}^{-1}\vec{\xi} \qquad \mathbf{R}^{-1} = \begin{pmatrix} | & | & \\ \hat{a}_1 & \hat{a}_2 & \dots \\ | & | & \end{pmatrix} \qquad \mathbf{R} = \left(\mathbf{R}^{-1}\right)^T \mathbf{M}$

  tensors : $\quad \mathbf{M}^{\xi} = \left(\mathbf{R}^{-1}\right)^T \mathbf{M}\mathbf{R}^{-1} \quad \rightarrow \quad \mathbf{M}^{\xi}_{mn} = \delta_{mn} \quad \& \quad \mathbf{K}^{\xi}_{mn} = \omega_m^2 \delta_{mn}$

  inhomogeneous EOM : $\quad \mathbf{M}\ddot{\vec{q}} + \mathbf{K}\vec{q} = \vec{F}$ in q-space $\quad \rightarrow \quad \mathbf{M}^{\xi}\ddot{\vec{\xi}} + \mathbf{K}^{\xi}\vec{\xi} = \left(\mathbf{R}^{-1}\right)^T \vec{F}$ in ξ-space

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Fourier Series as an Inner Product Space

- **Space** : $|f\rangle \equiv$ **τ-periodic functions** $f(t)$ that are periodic over $t = [-\tau/2 \rightarrow \tau/2]$, with $\omega = 2\pi/\tau$
  - **Basis #1** : $|n\rangle \equiv \begin{cases} \sin(n\omega t) & n = 1,\dots,\infty \\ 1/\sqrt{2} & n = 0 \\ \cos(n\omega t) & n = -1,\dots,-\infty \end{cases}$
  - **Inner Product #1** : $\langle g | f \rangle \equiv \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} g(t) f(t) dt$
  - **Basis #2** : $|n\rangle = e^{in\omega t}$
  - **Inner Product #2** : $\langle \tilde{g} | \tilde{f} \rangle \equiv \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} g^*(t) f(t) dt$

➔ **Basis is Orthonormal** : $\langle n | m \rangle = \delta_{nm}$

➔ **Completeness** : any $|f\rangle = \sum_{n=-\infty}^{+\infty} |n\rangle\langle n | f \rangle$

Lagrangian Mech from 325

* Gen. coord q_i must be indep

$$S \equiv \int_{t_1,\vec{q}_1}^{t_1,\vec{q}_1} dt\, L(q_i, \dot{q}_i, t) \qquad \delta S = 0 \rightarrow \frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) \text{ for each } q_i$$

$H \equiv \dot{q}_i(\partial L / \partial \dot{q}_i) - L$ conserved when $\partial L / dt = 0$

Principle of Least Action :

$$L = T - U \ \rightarrow \ \delta S = 0 \ @ \ \text{true } \{q_i(t)\}$$

Gen. force $Q_i \equiv \frac{\partial L}{\partial q_i}$, momentum $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$

$H \equiv p_i \dot{q}_i - L$ equals $T+U$ when $\vec{r}_a = \vec{r}_a(q_i)$