All you need for GR calculations is the Schwarzschild metric. The <u>Hamiltonian</u> = the particle's total energy (rest + kinetic + potential energy) is also useful because it is conserved. You derived it on the last homework for <u>radial motion</u>, so you can use it without proof this week:

$$d\tau^2 = dt^2\left(1 - \frac{2M}{r}\right) - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\phi^2 \quad \text{and} \quad e = \frac{dt}{d\tau}\left(1 - \frac{2M}{r}\right) \text{ for radial motion}$$

where $e = E_{\text{TOTAL}} / mc^2$ in SI units

Problem 1 : Time of Radial Fall

Alice is in a spaceship of mass m and constant energy e that falls under gravity (no engines) toward the center of a spherical mass M along a radial path. Last week, you used the energy formula and a bit of algebra to obtain

the Bookkeeper speed of such a radially-falling mass: $\dot{r} = -\left(1 - \frac{2M}{r}\right)\sqrt{1 - \left(\frac{1}{e}\right)^2\left(1 - \frac{2M}{r}\right)}$. To simplify your

calculations, let's use $e = 1$.

(a) What does $e = 1$ tell us about the spaceship's initial conditions?

(b) How much Bookkeeper time does it take for the spaceship to fall to the black hole's event horizon? Calculate a formula for $\bar{t} \equiv t / M$ as a function of $\bar{r} \equiv r / M$. (Just start calculating t and a function of r, then as you construct your integral, notice how M can be completely removed from the problem by rescaling t and r.) It requires a nasty integral, but one that wolframalpha can do just fine! To interpret your result, let's set our clock to $t = 0$ at somewhere finite, say at $r = 5$M. Plot $\bar{t}(r) - \bar{t}(r = 5M)$ versus $\bar{r}$ over the range $\bar{r} = 2$ to 8.

(c) How much proper time does it take for the spaceship to fall to the black hole's event horizon? Calculate a formula for $\bar{\tau} \equiv \tau / M$ as a function of $\bar{r} \equiv r / M$, then plot $\bar{\tau}(r) - \bar{\tau}(r = 5M)$ versus $\bar{r}$ again over the range $\bar{r} = 2$ to 8. Finally, <u>in words</u>: what exactly does the proper time difference $\tau(r=2\text{M}) - \tau(r=5\text{M})$ represent? i.e. What does it mean physically?

Problem 2 : Rolling Coin   *qual problem*

We must return to rotating systems in order to complete a *classic* qual exam problem. Shown at right is a coin, which we treat as a uniform disk of radius a and negligible thickness whose edge rolls without slipping on a table. The center of the coin moves in a circle of radius b when the axis of the coin is tilted at an angle θ with respect to the vertical. Find the angular velocity Ω of the center of mass of the coin (in the frame of the table).

Note: Don't assume that we left this problem until now because the Euler angles (θ,ϕ,ψ) are required. They might be helpful but they are not required. You might alternatively find the Euler *equations* helpful. Or not. Just clarifying that Euler's *angles* and Euler's *equations* are different. Euler did a lot.

Hint: A common strategy for this one is to go to a rotating frame … but it's not the only strategy.
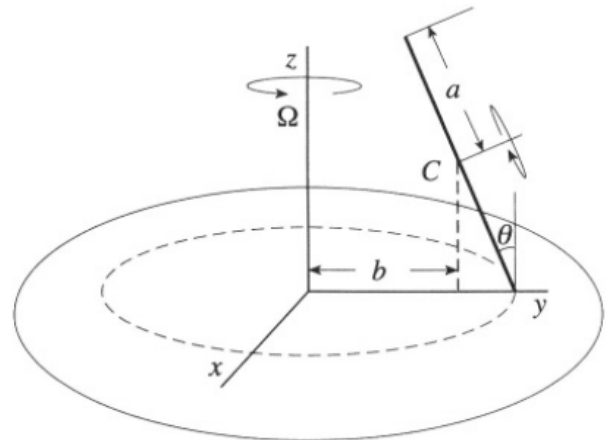
Problem 3 : The EM Lagrangian → Something You Need to Know

The Lagrangian for GR and SR is $L = -mc^2 d\tau / dt$ in SI units. This does <u>not</u> have the form $L = T - U \rightarrow$ not all Lagrangians do! (What you showed last week is that it *reduces* to T-U in the Newtonian limit of slow speed and weak gravity.) The Lagrangian that describes the motion of a non-relativistic particle of mass m and charge e in an electromagnetic field also does not have T–U form: it is

$$L_{EM} = \tfrac{1}{2}m\dot{\vec{r}}^2 - e(V - \dot{\vec{r}} \cdot \vec{A}) .$$

Here $\{V, \vec{A}\}$ are the {scalar,vector} potentials of E&M that give the fields via the relations $\vec{E} = -\vec{\nabla}V - \partial\vec{A}/\partial t$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. We are using "$e$" for the particle's charge since q could cause confusion with "generalized coordinate". Obviously we can write this as $L_{EM} = T - e(V - \dot{\vec{r}} \cdot \vec{A})$, which *looks* like T–U form, but it really isn't. Let's explore.

(●) Obviously you will be profoundly unhappy unless you verify for yourself that this unusual Lagrangian does indeed reproduce the E&M force law, $\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$! Use Cartesian coordinates (x,y,z) for the particle's position, then write down and expand the Euler-Lagrange equation for just one coordinate, x. If you can show that this equation reduces to $m\ddot{x} = F_x = e(E_x + \dot{y}B_z - \dot{z}B_y)$, it will be true for y and z as well. This derivation is done in Taylor section 7.9, but please do try it yourself.

(a) L_{EM} certainly *looks* like a typical Lagrangian of the form $L = T - U$, with a velocity-dependent potential energy $e(V - \dot{\vec{r}} \cdot \vec{A})$. However, this is *not* a potential energy and should even not be given the letter U … we can call it U^*. Why all the caution? The defining property of a potential energy U is that it describes a force via the standard relation $\vec{F} = -\vec{\nabla}U$. Well this one doesn't! Calculate the gradient of $U^* = e(V - \dot{\vec{r}} \cdot \vec{A})$ and show that $-\vec{\nabla}U^*$ is *not* the Coulomb-Lorentz force $\vec{F} = e(\vec{E} + \vec{v} \times \vec{B})$.

Hint 1: Perform your calculation in Cartesian components with index notation: $-\vec{\nabla}U^* = -\hat{x}_i \, \partial U^*/\partial x_i$

Hint 2: The easiest way to show that $-\vec{\nabla}U^*$ is *not* $e(\vec{E} + \vec{v} \times \vec{B})$ is to find one example that disproves it. Specifically, find a vector-potential $\vec{A}$ that produces *no magnetic field* (i.e. an $\vec{A}$ that has no curl) but does have some derivative: one that is sufficient to make an appearance in $-\vec{\nabla}U^*$ and cause it to *not* match the force $e\vec{E}$.

☞ **Take-away message**: L_{EM} is another example of a Lagrangian that does *not* follow the form $L = T - U$.

(b) A particle of mass m and charge e moves in a magnetic field produced by a constant current I flowing in an infinite straight wire that lies along the z-axis. This B-field is given by the potentials $A_s = A_\phi = V = 0$ and $A_z = -(\mu_0 I / 2\pi)\ln(s)$ in the cylindrical coordinate system (s,ϕ,z). Find the Lagrangian of the particle, note that ϕ and z are cyclic coordinates, and find the corresponding conserved quantities p_ϕ and p_z.

☞ **Crucial point**: The conserved p_z you just found is the generalized momentum conjugate to z… and it is NOT the z-component of mechanical momentum $(mv)_z$! → For this important E&M Lagrangian, we must be very careful to distinguish the two DIFFERENT concepts of "momentum":
  ● the <u>canonical momentum</u> $\vec{p}$ obtained from $p_i = \partial L / \partial \dot{x}_i$, and
  ● the <u>mechanical momentum</u> $m\vec{v}$ → Do NOT call this "p" for the rest of this problem, call it "mv" !

☞ **Jargon check**: The terms "<u>canonical</u> momentum", "<u>generalized</u> momentum", and "momentum <u>conjugate</u> to q_i" all mean exactly the same thing: $p_i \equiv \partial L / \partial \dot{q}_i$.

(c) Interpret the two conserved quantities you found by expressing them in terms of components of the following physical quantities: the particle's linear momentum $m\vec{v}$, the particle's angular momentum $\vec{l}$, its charge e, and/or $\vec{A}$ itself. Remember: to avoid massive confusion do <u>NOT</u> USE THE LETTER "p" for the combination "mv"! We are reserving "p" for canonical momentum.

(d),(e)-combined for clarity

To explore these two definitions of momentum further, consider the E&M Lagrangian and Hamiltonian for <u>any</u> electromagnetic potentials V and $\vec{A}$ (not just the ones from part b). Go to Cartesian coordinates $q_i = (x,y,z)$ and show that:

- the relationship between mechanical and canonical momentum is $m\vec{v} = \vec{p} - e\vec{A}$;

- the Hamiltonian for a charge e of mass m in an EM field is $H = \dfrac{1}{2m}(\vec{p} - e\vec{A})^2 + eV$.

☞ **The notorious substitution** : The relations you just found lead to a confusing "trick" that is invoked in other areas of physics, notably quantum mechanics. In various QM texts, you will see the author take a regular Hamiltonian, $H = T + U = p^2 / 2m + U$, then make this substitution: they replace $\vec{p}$ with $\vec{p} - e\vec{A}$ to "add electromagnetism" to their equations of motion. This substitution is often done with little or no explanation, or by saying it is "familiar from classical mechanics". (eye-roll) Now you know where it comes from … and you also know that the "p" in question is <u>canonical momentum</u>, NOT mechanical momentum.

(f) Finally, rewrite your Hamiltonian in this way: *remove* the canonical momentum $\vec{p}$ and *replace* it with the combination $m\vec{v} + e\vec{A}$. Since we *know* what $m\vec{v}$ is — it's the particle's actual, mechanical momentum — we can now interpret this Hamiltonian! Use your knowledge of the *work done by a magnetic field* to show that your E&M Hamiltonian is, in fact, the total (kinetic + potential) energy of a charge in an electromagnetic field.

☞ **A Vital Message**: The Hamiltonian $H_{EM} = (\vec{p} - e\vec{A})^2 / 2m + eV$ can lead to *enormous* confusion. (I have seen it happen, e.g. throughout a 2-week research workshop with numerous theorists in attendance.) The key point: **the "p" in this Hamiltonian is NOT the particle's momentum mv**! People constantly assume that it is, then go on to believe that the $-e\vec{A}$ term added to $\vec{p}$ is some sort of "field momentum". As you saw in (f), our Hamiltonian is nothing more than $H_{EM} = mv^2 / 2 + eV = T+U$ of the charged particle; it does not contain *any trace whatsoever* of the momentum or energy of the EM fields. If field energy plays a role, V and/or $\vec{A}$ will be time-dependent and lead to $\partial H / \partial t \neq 0$, i.e. a non-conserved Hamiltonian H. To restore conservation of energy, a different expression is needed that includes the energy in both the particles and the fields. Chapter 8 of Griffiths' "*Introduction to Electrodynamics*" contains an excellent discussion of energy and momentum conservation in E&M ... though not in Hamiltonian or Lagrangian form. To go further, you must study the Lagrangian formulation of field theories, and that is well beyond the scope of our course.