

Physics 326 – Homework #11

- $I_{ij} = \int dm (\delta_{ij}r^2 - r_i r_j) = \int dm \begin{pmatrix} y^2 + z^2 & -xy & -xz \\ \cdot & z^2 + x^2 & -yz \\ \cdot & \cdot & x^2 + y^2 \end{pmatrix}$
- $\vec{L}^{(B)} = \mathbf{I}^{(B)} \vec{\omega} \quad \forall \text{ body-fixed ref. pt. } B$
- Principal Axes \hat{e} : $\mathbf{I} \hat{e} = \lambda \hat{e}$ • $T = \frac{1}{2} \vec{\omega} \cdot \vec{L} = \frac{1}{2} \vec{\omega}^T \mathbf{I} \vec{\omega}$ • $\mathbf{I}^{(B)} = \mathbf{I}_{CM}^{(B)} + \mathbf{I}'$ • $\vec{\omega}_{1,3} = \vec{\omega}_{1,2} + \vec{\omega}_{2,3}$
- Euler's Equations : $\tau_1 = I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3$ • Free Symmetric Top : precession of $\vec{\omega}$ is
- $\vec{\tau} = \dot{\vec{L}} = \dot{\vec{L}} \Big|_{\text{within body}} + \vec{\omega} \times \vec{L}$ $\tau_2 = I_2 \dot{\omega}_2 + (I_1 - I_3) \omega_3 \omega_1$ $\vec{\Omega}^* = \left(\frac{I_3}{I_1} - 1 \right) \omega_3 \hat{e}_3 \text{ body, } \vec{\Omega} = \frac{\vec{L}}{I_1} \text{ lab;}$
- $\tau_3 = I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2$ $\vec{L}, \vec{\omega}, \hat{e}_3 \text{ always coplanar}$

You also proved several extremely useful symmetry theorems in Discussion 9; those are at your disposal too.

New stuff : the **Euler angles** (ϕ, θ, ψ) provide an excellent way of analyzing the general rotation of a rigid body because they can be readily interpreted. The main formulae and the key figure are :

$$\vec{\omega} = \dot{\phi} \hat{z} + \dot{\theta} \hat{e}'_2 + \dot{\psi} \hat{e}_3 \quad \text{directly from figure}$$

$$\vec{\omega} = -\dot{\phi} \sin \theta \hat{e}'_1 + \dot{\theta} \hat{e}'_2 + (\dot{\psi} + \dot{\phi} \cos \theta) \hat{e}_3 \quad \text{decomposit}^n \text{ into } \perp \text{ components}$$

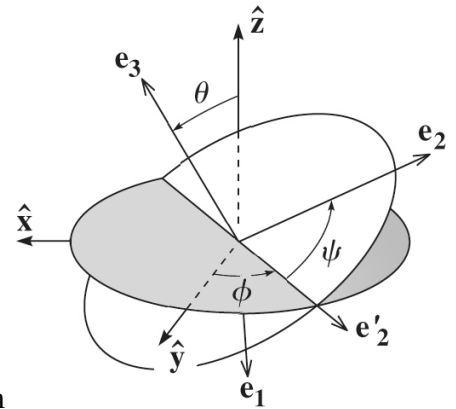
$$= \vec{\omega}_{12} + \vec{\omega}_3 \quad \text{using } \hat{z} = \cos \theta \hat{e}_3 - \sin \theta \hat{e}'_1$$

To obtain the angular momentum and kinetic energy, we as usual work with body-frame components, since the inertia tensor is diagonal in the body frame. That last decomposition of ω is perfect for this purpose!

For an axisymmetric body (i.e. one with $I_1 = I_2$) we write

$$\vec{L} = I_1 \vec{\omega}_{12} + I_3 \vec{\omega}_3 \quad \text{and} \quad 2T = \vec{\omega}^T \mathbf{I} \vec{\omega} = I_1 \omega_{12}^2 + I_3 \omega_3^2$$

as usual, then substitute in the Euler-angle expressions for $\vec{\omega}_{12}$ and $\vec{\omega}_3$ from

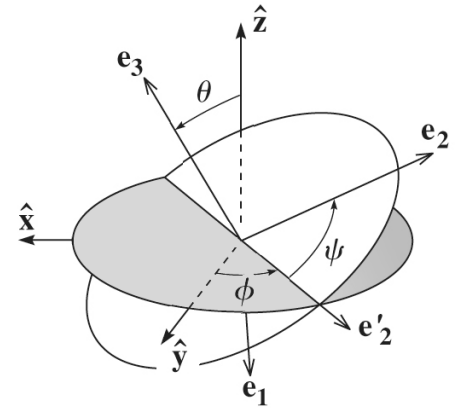


Problem 1 : Adding a Mountain to a Spherical Planet

Imagine that this world is a perfectly rigid uniform sphere (no equatorial bulge!) and is spinning about its usual axis at its usual rate. A huge mountain of mass 10^{-8} earth masses is now added at colatitude 60° , where “colatitude” means “angle with respect to the north pole”. (The Earth-geography latitude of this mountain would thus be 30° N.) The mountain ruins the earth’s perfect symmetry and so causes the Earth to begin free precession, like any other axisymmetric body. Assuming that no torques affect the Earth’s motion, how long will it take the rotational North Pole (the spot where $\vec{\omega}$ punches through the Earth’s surface) to move 60 km along the Earth’s surface from its current position? Take the Earth’s radius to be 6400 km. Note: the input values you’ve been given have been specified to at most *two significant digits*, so it makes *no* sense to obtain your answer to any greater level of precision than that. ☺

Problem 2 : Spinning Top on a Table

Euler's angles provide us with perfect generalized coordinates for a rigid body, and we can use them to build a Lagrangian. Hurray! Let's analyze the motion of an axisymmetric top spinning on a table in uniform gravity g . Put the origin at the location of the fixed pivot point on the table, define \hat{z} to point upwards and \hat{e}_3 to point along the top's axis of symmetry, and let R be the distance from the pivot to the CM of the top. We also know the moments of inertia I_3 and $I_1=I_2$ of the top and its total mass M . Our task is to figure out the top's general motion.



(a) Write down the Lagrangian $\mathcal{L}(\phi, \theta, \psi, \dot{\phi}, \dot{\theta}, \dot{\psi}) = T - U$ for this system.

(b) Your Lagrangian should have two cyclic coordinates: ψ and ϕ .

These give constants of motion, which is great!

First, find the constant generalized momentum $p_\psi \equiv \partial \mathcal{L} / \partial \dot{\psi}$ associated with the ψ coordinate.

(c) Generalized momenta associated with angles are usually some form of angular momentum. p_ψ is no exception. Which component of angular momentum is it?

You have a lot of choices: $L_x, L_y, L_z, L_1, L_2, L_3, L_1', L_2', \dots$ but it *is* one of these. ☺ You must *prove* your choice, of course!

(d) Now find the constant generalized momentum $p_\phi \equiv \partial \mathcal{L} / \partial \dot{\phi}$ associated with the ϕ coordinate. Once you have your expression, simplify it a bit using the constant $\omega_3 = \dot{\psi} + \dot{\phi} \cos \theta$ you obtained from (b),(c).

(e) What is the significance of this p_ϕ ? Again it is one of the components of \vec{L} , but it is not so easy to pick out. Take a guess: p_ϕ is associated with the motion $\dot{\phi}$ around the z -axis ... so it is probably L_z . Start with the general expression $\vec{L} = I_1(-\dot{\phi} \sin \theta \hat{e}'_1 + \dot{\theta} \hat{e}'_2) + I_3 \omega_3 \hat{e}_3$, calculate $L_z = \vec{L} \cdot \hat{z}$, and show that indeed $p_\phi = L_z$.

(f) Our third and final EOM is the one associated with θ . Write it down.

(g) This 2nd order equation is not trivial to solve. Let's take a special case: $\theta = \text{constant}$. This is a guess ... try it and see if it works. What conclusion do you reach for the time-dependence of $\dot{\phi}$ and $\dot{\psi}$?

(h) Let's introduce the label Ω for the constant $\dot{\phi}$. Knowing only that θ , $\dot{\phi}$, and $\dot{\psi}$ are constant, describe the motion of the top and the significance of Ω in that motion.

(i) Now consider the further special case where ω_3 is very big, i.e. far larger than any other quantity with the same dimensions). This is the usual situation when you spin a top: the rotation $\vec{\omega}$ you give it is as close to the \hat{e}_3 symmetry axis as possible, so $\omega \approx \omega_3 \gg \omega_1, \omega_2$. Using your EOM from part (g), solve for the constant precession frequency $\Omega \equiv \dot{\phi}$. You will obtain a quadratic equation with two solutions: one large, and one small. Use the $\omega_3 \gg$ everything approximation to find these two solutions.

$$\vec{\omega} = \dot{\phi} \hat{z} + \dot{\theta} \hat{e}'_2 + \dot{\psi} \hat{e}_3$$

$$\hat{z} = \cos \theta \hat{e}_3 - \sin \theta \hat{e}'_1$$

$$\begin{aligned} \vec{\omega} &= -\dot{\phi} \sin \theta \hat{e}'_1 + \dot{\theta} \hat{e}'_2 + (\dot{\psi} + \dot{\phi} \cos \theta) \hat{e}_3 \\ &= \vec{\omega}_{12} + \vec{\omega}_3 \end{aligned}$$

Problem 3 : Horizontal Spinning Top

The spinning top in the previous problem was axisymmetric, secured to a fixed pivot at exactly one point, and under the influence of uniform gravity g . Let's construct the exact same situation but raise the pivot point above the table. For example, we can nail a vertical post to the table, then attach the end of the top that was previously on the table to a pivot at the top of the post. In this way, we can spin the top so that its axis of symmetry is always horizontal. At $t = 0$, we hold the top's axis horizontal to the ground, then we spin the top around its axis of symmetry with angular speed $\vec{\omega} = \omega_0 \hat{e}_3$. Remarkably, no matter how slow the initial spin ω_0 is, the top's axis remains perfectly horizontal, even with gravity pulling it downwards. Wow! The only thing that changes is the direction of the top's symmetry axis, which precesses. It is amazing that this happens, but of course someone filmed it : <https://www.youtube.com/watch?v=ty9QSiVC2g0> ; the experiment is 3:00 into the video.

Your task is to find the precession frequency of the top's symmetry axis in the lab frame, remembering to specify both its magnitude and direction. Express your answer in terms of ω_0 and the given parameters of the top I_1, I_3, M , and/or $R =$ distance from top's pivot-point to its center-of-mass. I would like you to find the solution in two different ways :

(a) The precession frequency can be readily determined using the **Lagrangian EOMs** with **Euler angles** as generalized coordinates. This is the method you worked with in Problem 2. Using any of the relevant results you obtained in problem 2 without proof (since you already proved them!), determine the lab-frame precession vector of the horizontal spinning top.

(b) Now obtain the precession vector *without* using a Lagrangian. The tools you may use without proof are those at the top of page 1, i.e. the same formula set as for last week's homework. These tools are the **torque-based EOM** $\vec{\tau} = d\vec{L} / dt$ and the results we derived from it, e.g. Euler's equations and a couple of special-case results. This time, you cannot use any results from Problem 2; you must re-solve the problem from scratch without using a Lagrangian, so you can only use the given facts:

- the top's symmetry axis remains horizontal at all times
- at $t=0$, $\vec{\omega} = \omega_0 \hat{e}_3$.
- we know the values of ω_0, I_1, I_3, M , and R

Would you like some hints? The final page of the homework has a long essay to guide you. ☺

Problem 4 : Vertical Spinning Top that's Actually Stable

Take a cylindrically symmetric pencil, hold it vertically with the sharp end resting on a table, and spin it perfectly around its axis of symmetry with angular velocity ω_0 . Will it fall over? You cannot spin it *perfectly* around its axis of symmetry, so this is a stable-vs-unstable equilibrium question. Suppose the angle θ between the pencil and the vertical direction is arbitrarily small but not quite zero. Is it possible that a vertical rotation might actually be stable under tiny disturbances of θ and/or $\dot{\theta}$ around the equilibrium state $\{ \theta = 0, \dot{\theta} = 0 \}$?

If we don't give the pencil any spin ω_0 , this equilibrium state is for sure completely unstable (!) ... but amazingly, if ω_0 is large enough, you *can* get a stable vertical rotation.

(a) Calculate the minimum value of ω_0 needed to achieve such stable equilibrium in terms of these given parameters: $I_1=I_2, I_3, M, g$, and/or $d =$ distance from the end of the pencil to its center of mass.

(b) Calculate the minimum ω_0 in revolutions per second needed for the stable vertical rotation of :

- (b1) a solid cylindrical pencil of length 16 cm and radius 0.5 cm
- (b2) a hollow basketball of regulation radius 12 cm

on a table. Note : only 2 significant digits given therefore only 2 needed ☺ Also, you can lift whatever moments of inertia you like from Wikipedia's "List of Moments of Inertia" article.

GUIDANCE for Problem 3: Euler's equations are actually not the best way to go here. Unlike in the case of the Free Symmetric Top (FST), this problem has a non-zero torque. You can certainly solve Euler's equations with the torque to get all three components $\omega_1(t), \omega_2(t), \omega_3(t)$ in the body-fixed $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ coordinate system ... but then what? You are looking for the precession frequency of the top's axis in the LAB frame, so you must find some way to translate your $\omega_{1,2,3}$ solution into a lab-based motion. The way we did this for the FST was to rely heavily on \vec{L} , because in a torque-free situation, it is a *constant*. Well, this is not a torque-free situation. The problem is not *finding* $\vec{\omega}(t)$ from Euler's equations, the problem is *interpreting* it. Hmm.

The best procedure is to go back to the *source* of Euler's equations, $\vec{\tau} = d\vec{L} / dt$ and $\vec{L} = I_i \vec{\omega}_i$, and before you do anything else, **build an $\vec{\omega}$ that you can interpret**. To do so, consider all possible motions of the object in this problem (they are actually fairly limited), then use the **angular-velocity addition theorem** $\vec{\omega}_{a,c} = \vec{\omega}_{a,b} + \vec{\omega}_{b,c}$ to express $\vec{\omega}$ in terms of the precession vector you want and the other possible rotations. Remember that a, b, c are FRAMES, namely measuring environments where a given pair of vectors are considered fixed (see footnote¹). To inspire your thinking, here are some examples we have already seen of the ω -addition theorem in use; the FRAMES involved are in CAPS:

$$\begin{aligned} \text{In all cases,} & \quad \vec{\omega} = \text{BODY } \{ \hat{e}_3, \hat{e}_1 \} \text{ with respect to LAB } \{ \hat{z}, \hat{x} \} \\ \text{Free Symmetric Top : } & \vec{\omega} = \vec{\Omega} - \vec{\Omega}^* \quad -\vec{\Omega}^* = \text{BODY } \{ \hat{e}_3, \hat{e}_1 \} \text{ with respect to PLANE } \{ \hat{\omega}, \hat{e}_3, \hat{L} \} \\ & \quad \vec{\Omega} = \text{PLANE } \{ \hat{\omega}, \hat{e}_3, \hat{L} \} \text{ with respect to LAB } \{ \hat{x}, \hat{z} \equiv \hat{L} \} \\ \text{Euler Angles : } & \vec{\omega} = \dot{\phi} \hat{z} + \dot{\theta} \hat{e}'_2 + \dot{\psi} \hat{e}_3 \quad \dot{\psi} \hat{e}_3 = \text{BODY } \{ \hat{e}_3, \hat{e}_1 \} \text{ with respect to NODELINE-e3 } \{ \hat{e}_3, \hat{N} \equiv \hat{e}'_2 \} \\ & \quad \dot{\theta} \hat{e}'_2 = \text{NODELINE-e3 } \{ \hat{e}_3, \hat{N} \} \text{ w.r.t. NODELINE-z } \{ \hat{z}, \hat{N} \} \\ & \quad \dot{\phi} \hat{z} = \text{NODELINE-z } \{ \hat{z}, \hat{N} \} \text{ with respect to LAB } \{ \hat{z}, \hat{x} \} \end{aligned}$$

(Notice how in each “A with respect to B” statement, the frames A and B always share a vector in common? They *must*: that common vector is the direction of $\vec{\omega}_{A,B}$, which is necessarily stationary in both frames. Think about it and play with objects for a minute ... it will be clear ☺.) Once you have expressed $\vec{\omega}$ as the sum of <unknown precession vector that you want> + <other possible rotation vectors> in this fashion, build \vec{L} using the given principal moments $I_1=I_2$ and I_3 , then evaluate both sides of the torque-based EOM $\vec{\tau} = d\vec{L} / dt$. Euler's equations are nothing more than

$$\vec{\tau} = \dot{\vec{L}} = \vec{\omega} \times \vec{L} + \left. \dot{\vec{L}} \right|_{\text{within body}},$$

you are just doing that “by hand”, and using a form for $\vec{\omega}$ that is simplified by the constraints of the problem and can be readily interpreted.

¹ When I say a frame is a measuring environment where two given vectors are considered fixed, I mean “vector” as in “a physical arrow with a certain magnitude, direction, AND LOCATION”. I am being careless here since a vector like \hat{x} doesn't actually have a LOCATION, only a magnitude and a direction. That means I can create two frames where a given set of unit vectors $\{\hat{x}, \hat{y}\}$ (and therefore their cross-product \hat{z}) are fixed ... and, OOPS, these two frames might be moving at *different speeds* relative to each other. To avoid this, we must define a frame as “an environment where two given vectors AND A GIVEN POINT are fixed”. That will distinguish all rotating *and* boosted frames from each other. Nice. In the context of rigid body rotations (what we are doing now), no boosting is going on at all, so there is no need to keep mentioning that the frames in question also include a defining fixed point. All of our frames and coordinate systems are assumed to share a common origin. I just wanted to clarify that that is implied.