Physics 326 – Homework #10 due Friday, 1 pm

Here's a summary of all our formulae so far related to the inertia tensor and rigid body rotation :

- $I_{ij} = \int dm\, (\delta_{ij} r^2 - r_i r_j) = \int dm \begin{pmatrix} y^2+z^2 & -xy & -xz \\ \cdot & z^2+x^2 & -yz \\ \cdot & \cdot & x^2+y^2 \end{pmatrix}$
- $\vec{L}^{(B)} = \mathbf{I}^{(B)}\vec{\omega} \quad \forall$ body-fixed ref. pt. B
- Principal Axes $\hat{e}$: $\mathbf{I}\,\hat{e} = \lambda\,\hat{e}$
- $T = \dfrac{1}{2}\vec{\omega}\cdot\vec{L} = \dfrac{1}{2}\vec{\omega}^T\mathbf{I}\vec{\omega}$
- $\mathbf{I}^{(B)} = \mathbf{I}^{(B)}_{CM} + \mathbf{I}'$
- Euler's Equations :

$$\vec{\tau} = \dot{\vec{L}} = \dot{\vec{L}}\Big|_{\substack{\text{within}\\ \text{body}}} + \vec{\omega}\times\vec{L} \qquad \begin{aligned} \tau_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \\ \tau_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 \\ \tau_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 \end{aligned}$$

- Free Symmetric Top : precession of $\vec{\omega}$ is

$$\vec{\Omega}^* = \left(\frac{I_3}{I_1} - 1\right)\omega_3\,\hat{e}_3 \text{ body}, \quad \vec{\Omega} = \frac{\vec{L}}{I_1} \text{ lab;}$$

$\vec{L}, \vec{\omega}, \hat{e}_3$ always coplanar

You also proved several extremely useful symmetry theorems in Discussion 9; those are at your disposal too. The Free Symmetric Top part (last section of formulae above) starts in Lec 11A.

Problem 1 : Practice with Euler's Equations

(a) A rigid body is rotating freely, subject to zero torque. Use Euler's equations to prove that the magnitude of the angular momentum is constant. Hint: you can just show L^2 is constant, and L^2 has a very nice form in the body-system used for Euler's equations! Write down its derivative, dL^2/dt, then manipulate Euler's equations (linear combination!) to build that dL^2/dt expression … it comes out really nicely. ☺

IMPORTANT: For this and all other parts of this problem, please use Euler's three equations to solve them, even if you could solve them some other way, e.g. by directly using the "master" equation $\vec{\tau} = d\vec{L}/dt$ on which Euler's equations are based. To be exact, please don't use the vector equation $\vec{\tau} = d\vec{L}/dt$ at all : the entire point of this problem is to become familiar with the structure of Euler's equations. All parts of this problem can be solved by taking linear combinations of Euler's equations (or using just one of them) to construct the quantity you are studying in each part.

(b) In much the same way, show that the kinetic energy of rotation, $T_{rot} = \frac{1}{2}\vec{L}\cdot\vec{\omega}$, is constant under zero torque.

(c) Consider a lamina rotating freely (no torques) about a point O that is within the lamina. Use Euler's equations to show that the component of $\vec{\omega}$ in the plane of the lamina has constant magnitude.
(i.e. If you choose $\hat{e}_3$ as perpendicular to the lamina, you must show that the time derivative of $\omega_1^2 + \omega_2^2$ is zero.)
Hint: A pure lamina is completely flat : it has *no size* in the direction perpendicular to its surface. This causes an additional simplification in the inertia tensor beyond certain off-diagonal elements going to zero → it imposes a strict relationship between the diagonal elements, i.e., one of them can be written in terms of the other two. You'll need this relationship; it's easy to figure out.

(d) Consider an axisymmetric object rotating freely (i.e. no torques) about a point O on its axis of symmetry. What do Euler's equations tell us about the time-dependence of the component of $\vec{\omega}$ along the object's axis of symmetry?

Problem 2 : Small Oscillations of a Spinning Book

We used Euler's equations in lecture to analyze the rotational stability of a book, and we checked it with a demo. Your turn to analyze the demo! A book of uniform density and dimensions (a=30 cm) × (b=20 cm) × (c=3 cm) is held shut with a rubber band. You throw the book into the air spinning at 180 rpm (revolutions per minute) about an axis that is *very close* to the book's shortest symmetry axis (i.e. the axis parallel to the shortest dimension of the book). What is the frequency of small oscillations of the book's axis of rotation in the book's body-frame? Give your answer in rpm, so you don't have to convert this awkward unit into anything else.

FYI : The theorem we proved that an asymmetric object rotates stably around the principal axes with the largest and smallest moments of inertia and unstably around the PA with moment of inertia in between these extremes goes by various names including the "Intermediate Axis Theorem" and the "Tennis Racket Theorem". A tennis racket, like a narrow book, is a good demonstration object for this theorem because its three principal moments are very different. Another outstanding demonstration object is your cellphone. Try flipping it around its three principal axes … you will see that it rotates nicely around the long axis and around the axis normal to the screen, but if you flip it around the short axis, it will always do a funky gymnastic *twist* in the air before it returns to your hand. Oh and don't break your phone.

<<< The evolving tensor-transformation problem that was here previously will appear next week. If you worked on it already, hang on to your work! >>>

Problem 3 : Angles for a Free Symmetric Top

In our study of a torque-free symmetric top, we found the exceedingly important relation that the vectors $\vec{L}$, $\vec{\omega}$, and $\hat{e}_3$ always remain coplanar. (Recall that $\hat{e}_3$ is the axis of symmetry of the top.) This coplanarity provides a crucial link between the body frame, where $\hat{e}_3$ is fixed, and the lab frame, where $\vec{L}$ is fixed. In addition, the angles between these three vectors remain constant throughout the object's motion.

Just FYI: for basically all "free top" problems, the quantities you must be given to make the system solvable are

- the top's principal moments I_i, or enough information about the top to calculate them
- some information about $\vec{\omega}$, e.g. the components ω_3 and $\omega_{12} = |\omega_1\hat{e}_1 + \omega_2\hat{e}_2|$ (which are constants of motion for an axisymmetric top), the magnitude ω and some angle, or some initial value $\vec{\omega}\big|_{t=0}$

(a) Calculate the angle α between the vectors $\vec{L}$ and $\hat{e}_3$ in terms of I_1, I_3, ω_{12}, and/or ω_3.

(b) Do the same for the angle β between the vectors $\vec{\omega}$ and $\hat{e}_3$.

(c) A thin, flat, uniform circular disc is thrown into the air so that it spins with angular velocity ω about an axis that makes an angle β with the symmetry axis of the disc. What is the precession frequency of the disc's symmetry axis around the angular momentum vector, as seen in the lab frame? Amazingly, the answer depends only on ω and $\sin\beta$.

Problem 4 : The Space Station from 2001

An axially symmetric space station (e.g. the torus depicted in the movie "2001") rotates in empty space. It has rockets mounted symmetrically on opposite sides. The rockets fire continuously so as to exert a constant torque $\vec{\tau} = \tau\,\hat{e}_3$ around the station's axis of symmetry, $\hat{e}_3$. The principal moments I_1 and I_3 of the station are known.

The station's rotation is *not* aligned with its symmetry axis : at time $t = 0$, the rotation vector is $\vec{\omega}\big|_{t=0} = \omega_{20}\hat{e}_2 + \omega_{30}\hat{e}_3$, where ω_{20} and ω_{30} are constants. Solve Euler's equations exactly for $\vec{\omega}(t)$ in the body coordinate system using this initial condition. If you need it, a math hint is provided at the end about how to solve the coupled differential equations you will encounter.

Math Hint for Problem 4 : The Space Station

You will encounter coupled equations of this type: $\begin{aligned}\dot{f}_1(t) &= -K(t)f_2(t)\\ \dot{f}_2(t) &= \quad K(t)f_1(t)\end{aligned}$.

If that coefficient $K(t)$ was a *constant*, K, you would immediately know the form of the functions: one of them is a sine and one of them is a cosine. They may have some overall phase shift or some amplitudes of common magnitude to satisfy the boundary conditions, e.g. $-6\sin(\omega t - 45°)$ and $6\cos(\omega t - 45°)$ … but whatever the details, you know f_1 and f_2 are sinusoidal functions of time that are 90° out of phase with each other. There is no other pair of functions that will give you "derivative of f_1 is –*blah* f_2 and derivative of f_2 is +*blah* f_1". The only unfamiliar aspect of problem 3(a) is that *blah* is a function of time, not a constant. Well you can *still* solve the equations by guessing well. Will a sine and a cosine still work? Absolutely : even when *blah* is time-dependent, there is still no other pair of functions that gives you "derivative of f_1 is –*blah* f_2 and derivative of f_2 is +*blah* f_1". You are accustomed to the solution forms $A\sin(\omega t + \phi)$ and $A\cos(\omega t + \phi)$… you just have to rethink them a little bit. You need some additional time-dependence somewhere, to accommodate that $K(t)$ coefficient ... where shall we put it? How about in the argument of the sinusoidal functions? ➔ <u>ωt is a bit too simple</u>, that's all, so try replacing it with some unknown function of time, $g(t)$: try $A\sin[g(t)]$ instead of $A\sin[\omega t]$. Plug forms like that for f_1 and f_2 into your differential equations and you will quickly see what $g(t)$ has to be.