Physics 326 – Homework #8 **due FRIDAY, 1 pm**

Our 2-body central force formula-set is now complete: we have added (1) <u>repulsive</u> $1/r^2$ (Kepler) forces with negative force-constants γ, and (2) relations needed for <u>scattering</u> problems, namely formulae for the scattering angle θ and impact parameter b for unbounded Kepler orbits as well as general cross-section formulae.

- **Coordinates & Reduced Mass** : $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$, $\mu = \frac{m_1 m_2}{M}$
- **Centrifugal** force & PE : $\vec{F}_{cf} = \frac{L^2}{\mu r^3}\hat{r}$, $U_{cf} = \frac{L^2}{2\mu r^2}$, effective radial $U^* = U + U_{\text{cf}}$
- **Angular EOM** : $\dot{\phi} = \frac{L}{\mu r^2}$ • **Radial EOMs** : $\mu\ddot{r} = F(r) + F_{\text{cf}}(r)$, $E = T + U(r) = \frac{1}{2}\mu\dot{r}^2 + U_{\text{cf}}(r) + U(r)$
- **Path Equation** : $u(\phi) \equiv 1/r(\phi) \quad \rightarrow \quad u'' + u = -\frac{\mu F(1/u)}{L^2 u^2}$ and $u' = -\frac{\mu \dot{r}}{L}$

- **Conics** : With (r,ϕ) centered on a <u>focal point</u> and E ≡ Ellipse, H ≡ Hyperbola

$$\frac{1}{r} = \frac{a}{b^2}(\pm 1 + e\cos\phi) \text{ with } \begin{cases} +\text{: E or H-near-branch} \\ -\text{: H-far-branch} \end{cases} , \qquad e = \frac{c}{a} = \frac{\sqrt{a^2 \mp b^2}}{a} \text{ with } \begin{cases} -\text{: E} \\ +\text{: H} \end{cases}$$

- **Kepler Orbits** $F = -\frac{\gamma}{r^2}$: $r(\phi) = \frac{r_0}{\text{sgn}[\gamma] + e\cos\phi}$ with $r_0 = \frac{L^2}{\mu|\gamma|} = \frac{b^2}{a} = a\left|1-e^2\right|$, $E = \mp\frac{|\gamma|}{2a} = \frac{|\gamma|(e^2-1)}{2r_0}$

  Bounded orbits: $\tau^2 = \frac{4\pi^2\mu}{\gamma}a^3$, $r_0 = \frac{2 r_{\text{min}} r_{\text{max}}}{r_{\text{min}} + r_{\text{max}}}$, $e = \frac{r_{\text{max}} - r_{\text{min}}}{r_{\text{max}} + r_{\text{min}}}$

  Unbounded orbits: scattering angle $\theta = \pi - 2\alpha$ with $\tan\alpha = \frac{b}{a}$, impact parameter b = semi-minor axis b ☺

- **XSec** : $d\Omega \equiv \frac{dA}{r^2} = \sin\theta\, d\theta\, d\phi$, $\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$ with θ = scattering angle • **Lumi**: $\mathcal{L} = n_A N_e$, $\frac{dN_{\text{ev}}}{dt} = \mathcal{L}\sigma$

- **Earth Data**: radius of the earth is $R_\oplus = 6.4 \times 10^6$ m; all appearances of the earth's mass $M_\oplus$ will be in the combination $GM_\oplus$ which is equal to $gR_\oplus^2$; use that and the familiar value $g = 9.8\ \text{m/s}^2$.
- **Atomic Data**: Gas @ STP has $N_{\text{Avog}} = 6.02\times10^{23}$ molecules in 22.4 liters of volume; 1 amu = 1.66×10^{-27} kg.

Practice Problem 0 [NOT FOR POINTS]: The Rutherford Cross Section

We have all the tools we need to derive the most famous cross section in the world: the **Rutherford XSec** $d\sigma / d\Omega$ for the non-relativistic scattering of two charged particles. Ernest Rutherford used this calculation to analyze the 1911 scattering experiment of Geiger and Marsden and deduce that the positive charge in the atom is *not* smeared uniformly within the atom but *concentrated in a tiny volume*. This was the <u>discovery of the atomic nucleus</u>. On to our derivation! We give the beam particle a charge q and the target particle a charge Q; the force between them is then $F = kqQ/r^2$. We also assume that the target particle's mass is so much greater than the beam particle's mass ($M \gg m$) that the target can be treated as fixed. You have all the tools you need to show that this famous cross-section is

$$\frac{d\sigma}{d\Omega} = \left[\frac{kQq}{4E\sin^2(\theta/2)} \right]^2 \qquad \textbf{(Rutherford Cross Section)}.$$

If you need them, steps are in the footnote [1]. Also you will find these trig relations helpful:

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta), \qquad \frac{d}{d\theta}\left[\cot(\theta/2)\right] = -\frac{1}{2\sin^2(\theta/2)}, \qquad \sin\theta = 2\sin(\theta/2)\cos(\theta/2)$$

Problem 1 : Scattering Quickie #1

The total cross section for scattering a certain nuclear particle (call it "the beam particle") by a nitrogen nucleus is 0.5 barns where **1 barn = 10^{-28} m^2**. A beam of 10^{11} of these particles per second are fired through a cloud chamber of length 10 cm containing nitrogen gas at STP. Recall from chemistry that nitrogen gas is written N_2 because one molecule contains two atoms, and note that the cross section we are given here is for one nitrogen nucleus.

(a) Calculate the luminosity of this experiment in cm^{-2} s^{-1}. (Use the ideal gas law)

(b) How many particles are scattered per second? (Ignore any scattering from atomic electrons.)

(c) How long would the experiment have to run to collect a total luminosity of 1 pb^{-1}?

Problem 2 : Scattering Quickie #2

The cross section for scattering a certain nuclear particle from a copper nucleus is 2.0 barns. If 10^9 beam particles are fired through a copper foil of thickness 10 μm, how many of them are scattered? You will need these values: the density of copper is 8.9 g/cm^3 and the atomic mass of copper is 63.5. Recall: the atomic mass gives the mass of an element in amu = atomic mass units where **1 amu = 1.66 × 10^{-27} kg** ≈ the mass of a proton. (The mass of a proton is 1.67 × 10^{-27} kg; the strict definition of an amu is the mass of the ^{12}C nucleus divided by 12, which is not exactly the same due to the binding energy of ^{12}C.)

Problem 3 : Scattering Quickie #3

(a) Using the Rutherford cross section formula from Problem 0, calculate the differential cross section for scattering 6.5 MeV α-particles (^{4}He nuclei) off a silver nucleus (^{108}Ag) at a scattering angle of 120°. Express your result in barns/sr. Since the target nucleus (atomic mass 108) is so much heavier than the beam nucleus (atomic mass 4), treat the target as infinitely more massive than the beam.

(b) If a total of 10^{10} α-particles impinge on a silver foil of thickness 1 μm and we detect the scattered alphas using a counter of area 0.1 mm^2 placed 2 cm from the target at a scattering angle of 120°, how many scattered alphas will we count? (Silver has a mass density of 10.5 g/cm^3 and an atomic mass of 108.)

---

[1] Here are the steps to take in your derivation. **Step 1: strategize** (A) Looking at our $d\sigma / d\Omega$ formula, we see that we need to relate impact parameter b to scattering angle θ → that's our main goal. (B) Looking at the answer we're trying to obtain, we see that the given parameters — the "things we know" — are going to be E, θ, and of course $kQq = |\gamma|$. So our "strategy box" looks like this: we know E, θ, $|\gamma|$ and we want b. **Step 2: get b in terms of θ** We have a $1/r^2$ force here, so we are in the world of Kepler orbits → go through that rich section of the formula sheet looking for the relations you need to determine <u>what you want</u> = b in terms of <u>what you know</u> = E, θ, $|\gamma|$. Result: $b = (|\gamma| / 2E)\cot(\theta/2)$. **Step 3: get dσ/dΩ** Now it's just plug-and-play using our differential cross section formula and the trig relations provided.

Problem 4 : Cross Section Calculation

A uniform flux of particles, each of mass m and initial speed v_0 , is incident upon a fixed scatterer (i.e. an very massive one with $M \gg m$) that exerts a repulsive radial force $\vec{F} = (m\gamma^2 / r^3)\,\hat{r}$. NOTE 1: This is not a $1/r^2$ force … which makes a significant portion of our formula collection irrelevant. NOTE 2: Feel free to use a computer (wolframalpha.com, fancy calculator, etc) to do the calculus for this problem as it's a bit nasty. ☺

(a) Calculate the differential cross section $d\sigma / d\Omega$ for this experiment as a function of the scattering angle θ.
HINT: Determining the scattering angle θ requires some thought. First, recall Discussion 6 Question 1, where you found the *apsidal angle* $\Delta\phi$ for a comet's unbounded trajectory. Make a quick sketch of what such an unbounded trajectory looks like, mark $\Delta\phi$ on your sketch, and consider: if we interpret the trajectory as a *scattering experiment*, how is the scattering angle θ related to $\Delta\phi$?

(b) Calculate the total back-scattering cross section, i.e. for scattering at angles greater than 90°.
NOTE: I'm sure your curiosity will impel you to also calculate the *forward*-scattering cross section. You will find it is infinite, which is not an error but exactly as it should be. Think about it and make sure that infinity makes sense (if it doesn't make sense, be sure to ask in class or at office hours! ☺)

To conclude our work on central forces and scattering, let's work a couple of problems that have appeared on Ph.D. qualifying exams. In a "qual" exam, please note that you do not get a formula sheet. This is one tiny reason why it is important to know the sequence of derivation that we've taken through each topic; the big important reason, of course, is to understand exactly where all our results come from and what assumptions they are based on, and to impose the strongest possible structure in your mind on all the physics you are learning. So, with the page 1 formula sheet in hand, on to a couple of qual problems about central forces and scattering.

Problem 5 : Central Force Orbit

(a) Find the central force that results in the following orbit for a particle: $r = a(1+\cos\phi)$ where a is a constant.

- Not part of the qual problem: try sketching the orbit … this weird shape is apparently called a "cardioid". ☺
- Hint: you must introduce some parameters that were not given to you; same thing for part (b). *Hard core!* ☺

(b) A particle of mass m is acted on by an attractive force whose potential is given by $U \sim r^{-4}$. Find the total cross section for capture of the particle coming from infinity with an initial velocity v_∞.

Problem 6 : Yukawa Force Orbit

A particle of mass m moves in a circle of radius R under the influence of a central attractive force $F = -\dfrac{K}{r^2} e^{-r/a}$.

(a) Determine the condition(s) on the constant a such that the circular motion will be stable against small radial perturbations.

(b) Compute the frequency of small radial oscillations about this circular motion.

For your information (has nothing to do with solving the problem): The **Yukawa potential** is a standard form of potential in subatomic physics. It describes a force mediated by the exchange of a spin-0 particle (a "boson") whose mass is $m = \hbar / ac$ in terms of the a parameter in that exponential $e^{-r/a}$. The Yukawa potential is usually written $U \sim -(1/r)e^{-mcr/\hbar}$ and appears in intro texts on nuclear physics. Using m as the mass of the pion = the lightest bound state you can make from quarks, the Yukawa one-pion-exchange potential (OPEP; yes it has an acronym) gives a pretty good description of the long-range part of the nucleon-nucleon interaction. The Yukawa potential doesn't *quite* match the force form in this problem, but close enough; I guess the problem's authors dropped the second term you get from $F = -dU/dr$ to shorten the problem. Oh and by the way, you may have heard that in the world of quantum field theory (= quantum mechanics + relativity), all forces are treated as being mediated by some exchange particle. For the electromagnetic force, the exchange boson is the photon … whose rest mass is ZERO … which gives a Yukawa potential of $U \sim -(1/r)$ = exactly the form of the Coulomb potential you know and love. ☺ For forces with massive exchange particles, such as the pion for the nucleon-nucleon force, that exponential factor $e^{-mcr/\hbar}$ serves to *cut off* the potential at large distances; the heavier the exchange particle, the shorter-range the force. Indeed, the nucleon-nucleon force *is* a short-range force. When I said above that the OPEP works for the "long-range" part of the force, I mean at most 10's of femtometers; after that, the nucleon-nucleon force is totally negligible compared with the electromagnetic force.