Physics 326 – Homework #7 due Friday, 1 pm

For today's homework, the **Formula Collection** you may use without proof is the entire set of formulae provided before the first problem of **Discussion 7**, including the two equations at the end for r_0 and e in terms of r_{min} and r_{max}. Everything else, you must derive here. As in discussion, use this **Earth Data**: radius of the earth is $R_\oplus = 6.4 \times 10^6$ m; all appearances of the earth's mass $M_\oplus$ will be in the combination $GM_\oplus$ which is equal to $gR_\oplus^2$; use that and the familiar value $g = 9.8$ m/s^2.

Problem 1 : Pluto to Saturn – Hohmann Transfer

(a) Before we even introduce this problem, calculate a super-useful general formula for the velocity v_{apse} of an object at an apsidal point of a closed Kepler orbit (i.e. any bound orbit resulting from a central force $\vec{F} = -\hat{r}\,\gamma / r^2$): obtain v_{apse} in terms of the corresponding apsidal distance r_{apse}, the parameter r_0 of the orbit, the force constant γ, and the reduced mass μ of the system. Once you have this formula, you may use it in all subsequent problems.

A research satellite is in Pluto's orbit and needs to be transferred to Saturn's orbit. Assume that the satellite is never close enough to Pluto or Saturn to be influenced by their gravitational fields — i.e. the planets Pluto and Saturn themselves play no role in this problem, only the radii of their orbits matter. The distance from the Sun to Pluto is about 40 A.U. while the distance from the Sun to Saturn is about 10 A.U. To save precious funds, the satellite controllers accomplish the switch using a **Hohmann transfer**. If you didn't get to the Hohmann transfer problem in Discussion 7, please go through it (solution is posted) before you start this problem.

(b) Calculate the thrust factors $\lambda \equiv v_{\text{after}} / v_{\text{before}}$ at the launch point from Pluto (λ_L) and at the rendezvous point with Saturn (λ_R). NOTE for this and future problems: Whenever you have a planet orbiting around a star, or a satellite around a planet, the orbiting mass m is so much smaller than the central mass that you may always assume that $\mu \approx m$ (unless otherwise specified, of course).

(c) Show that the satellite's final speed is twice its initial speed.

(d) Calculate the total travel time from launch at Pluto to rendezvous at Saturn.

Problem 2 : GSO = Geo-Synchronous Orbit

An earth satellite is in a circular orbit 250 km above the Earth's surface. Using the most fuel-efficient method available, NASA controllers fire the satellite's thrusters once in order to transfer it to an elliptical **geosynchronous orbit** (GSO) = an orbit whose period matches that of the Earth's rotation. The period of this orbit is thus 24 hours.[1]

(a) Calculate the velocity change, Δv, that is imparted to the satellite to achieve the desired orbit. Remember to include both the magnitude and sign of Δv.

(b) Calculate the apogee distance of the geosynchronous orbit. (There's a decent chance you already calculated it in part (a); if so, emblazon the result with a nice box and an arrow so the grader can find your work. ☺)

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[1] Actually, satellites in GSO, such as many communications and TV satellites, have an orbit of 23 hours 56 minutes = 1 **sidereal day**. This is the period of the Earth's orbit relative to the fixed stars, which provide a fixed inertial reference frame. 24 hours is the period of a **solar day** = the time it takes for the sun to return to the same position in the sky. Since the Earth orbits around the sun, these two periods are not exactly the same, but to simplify your calculator work, just take 24 hours for the period of a satellite in GSO.

Problem 3 : Escape!

(a) Calculate the **escape velocity** v_e of an object as a function of its distance r from the center of the Earth. Jargon check: "escape velocity" is the minimum velocity an object needs at its starting point in order to be able to make it to $r = \infty$ (i.e. to "escape" and become a "free particle") with its engines turned off.

(b) A spaceship is parked in an elliptic orbit around the Earth. The ship's crew have a long journey ahead of them and they must first escape the Earth's gravitational field. The crew wishes to escape from their elliptical orbit using the most fuel-efficient method possible. As we learned in Phys 325 : rocket motion, the amount of fuel used is proportional to the change Δv in the rocket's speed. The crew will apply one impulse from their engines to achieve escape velocity; the question is: at what point on their elliptical parking orbit should they fire their engines to achieve optimal fuel efficiency?

Hint 1: The optimal point is at one of the apses, but you must *show* that this is so, and you must determine which apse (perigee or apogee) is best. This problem is not as trivial as it might seem: in its parked orbit, the ship has the highest velocity at perigee, but it also has the highest *escape velocity* there since perigee puts the ship deeper into the Earth's potential well than at any other point on its orbit. There are thus two competing effects at work, making for an interesting optimization problem.

Hint 2: A function $f(x)$ has an extremum at points where $df/dx = 0$ … unless there is *no* point where $df/dx = 0$ within the *range of validity* of x. Then how do we optimize f? Consider this: f is the price of a hotel room for the GenCon50 on August 17, and x is the date in August. The hotels will keep increasing their prices as Aug. 17 approaches, so df/dx will be positive over the entire range of x from 1 to 17. How do you pick the date x that minimizes the price f when df/dx is never zero?? ;-)

Problem 4 : A Satellite Experiences a Drag Force

A satellite of mass m moves in the gravitational field of the Earth (mass $M \gg m$), but it is also subject to a linear drag force $\vec{F}_{\text{drag}} = -2\beta m\vec{v}$ where β is the usual positive drag constant.

(a) Show that the satellite's equations of motion can be reduced to the form:

$$\ddot{r} + 2\beta\dot{r} + \frac{GM}{r^2} - \frac{L_0^2 e^{-4\beta t}}{r^3} = 0 \qquad \text{and} \qquad r^2\dot{\phi} = L_0 e^{-2\beta t}$$

where L_0 is a constant that will be assumed to be positive. (If it's negative, the only thing impacted is the sign of $\dot{\phi}$, i.e. whether the satellite moves in the $+\phi$ or $-\phi$ direction.)

(b) Suppose that the drag force is small — i.e. that β is much smaller than all other quantities of the same units — and that the satellite begins in a circular orbit at $t = 0$. By neglecting the terms in $\dot{r}$ and $\ddot{r}$, find an approximate solution for $r(t)$ and $\dot{\phi}(t)$. Your solution should show that small resistance causes the orbit's radius to contract slowly; however, what happens to the speed of the satellite as time increases?

(c) In part (b) we neglected terms in $\dot{r}$ and $\ddot{r}$. This approximation is certainly justified for times near $t = 0$ since the orbit is initially circular, and a circular orbit has both $\dot{r} = 0$ and $\ddot{r} = 0$. Given the solution you obtained, can our neglect of these terms also be justified for times $t > 0$?

<<< One more question (very short but important) → turn the page! >>>

Problem 5 : Orbit of the Moon

The formulae on page 1 are all tools for obtaining information about one vector: the relative position vector $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ between two objects. We can obtain its components as functions of time, $r(t)$ and $\phi(t)$, and as functions of each other, $r(\phi)$, all by pretending that $\vec{r}$ describes a fake mass μ moving around a fake fixed source at the origin that produces a central potential $U(r)$.

That's a lot of fakery … What exactly is the connection between this fake, equivalent 1-body system and the actual, physical 2-body system m_1 and m_2? There are two connections:

- $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ IS the relative position between the two actual objects, and
- $U(r)$ IS the potential energy between the two actual objects

Nothing fake about those two quantities! To complete our return to reality, let's study the actual positions $\vec{r}_1$ and $\vec{r}_2$ of the objects in a 2-body system.

Take the Earth as object 1 and the Moon as object 2. The perigee distance is 363,000 km and the apogee distance is 405,000 km. (Remember, perigee means "closest approach to the Earth", apogee means "furthest separation from Earth". Also, when our objects are not points it is implicit that their "positions" are the positions of their centers.) Also, the mass of the Moon is 1.2% of the mass of the Earth: $m_2/m_1 = 0.012$. Using this information, describe the motion of of the Earth's center and the motion of the Moon's center in the **CM frame of the Earth-Moon system**. (No more fake system now! We are asking how the Earth and Moon actually move as viewed by an inertial observer sitting far out in space.)

(a) Convince yourself that the Earth and Moon both move along ellipses, and figure out the eccentricity of each ellipse.

(b) What lies at the focal point of the Earth's ellipse? (The Moon? The Earth-Moon CM? Nothing in particular? A chest with 5M EXP?) Similarly, what lies at the focal point of the Moon's ellipse?

(c) What is the semi-major axis, a_1, of the Earth's ellipse? (This tells you how much the Earth's center actually moves as the Earth and Moon orbit around each other.) Express a_1 as a percentage of the Earth's radius of 6,400 km … you might be surprised at the result. ☺