Physics 326 – Homework #4 due Friday 1 pm

<u>FORMULAE</u> : Inner Product Space description of normal modes, including Normal Coordinates

- **Space** : $|\vec{q}(t)\rangle \equiv$ all solutions of a particular linear oscillator system

- **Inner Product** : $\qquad \langle \vec{q}_1 | \vec{q}_2 \rangle \equiv \vec{q}_1^{\,T} \mathbf{M} \vec{q}_2 \qquad\qquad$ and associated **magnitude** : $|\vec{q}|^2 \equiv \langle \vec{q} | \vec{q} \rangle$

- **Basis** : $|\hat{a}_m\rangle$ of eigenvectors defined by $\mathbf{K}\vec{a}_m = \omega_m^2 \mathbf{M}\vec{a}_m$ and normalization $\hat{a}_m \equiv \vec{a}_m / |\vec{a}_m|$

- **Basis is Orthonormal** : $\langle \hat{a}_n | \hat{a}_m \rangle = \delta_{nm}$

- **Completeness** for $\vec{q}(t)$ $\quad$ and $\quad$ **Normal Coordinates** ξ_m :

  ξ_m is the **component** of $\vec{q}$ along mode m : $\quad |\vec{q}(t)\rangle = \sum_{\text{modes } m} |\hat{a}_m\rangle\langle \hat{a}_m | \vec{q}(t)\rangle \equiv \sum_m \hat{a}_m \, \xi_m(t) = \sum_m \hat{a}_m \, \tilde{A}_m e^{i\omega_m t}$

  ξ_m is **projected out** of $\vec{q}$ by inner product: $\quad \xi_m(t) = \langle \hat{a}_m | \vec{q}(t) \rangle \ = \tilde{A}_m e^{i\omega_m t} = A_m \cos(\omega_m t - \delta_m)$

- **Transformation** between q-space and ξ-space :

  vectors : $\qquad \vec{\xi} = \mathbf{R}\vec{q} \qquad \vec{q} = \mathbf{R}^{-1}\vec{\xi} \qquad \mathbf{R}^{-1} = \begin{pmatrix} | & | \\ \hat{a}_1 & \hat{a}_2 & \dots \\ | & | \end{pmatrix} \qquad \mathbf{R} = \left(\mathbf{R}^{-1}\right)^T \mathbf{M}$

  tensors : $\quad \mathbf{M}^\xi = \left(\mathbf{R}^{-1}\right)^T \mathbf{M}\mathbf{R}^{-1} \quad \rightarrow \quad \mathbf{M}_{mn}^\xi = \delta_{mn} \quad \& \quad \mathbf{K}_{mn}^\xi = \omega_m^2 \delta_{mn}$

  inhomogeneous EOM : $\quad \mathbf{M}\ddot{\vec{x}} + \mathbf{K}\vec{x} = \vec{F}$ in q-space $\quad \rightarrow \quad \mathbf{M}^\xi \ddot{\vec{\xi}} + \mathbf{K}^\xi \vec{\xi} = \left(\mathbf{R}^{-1}\right)^T \vec{F}$ in ξ-space

<u>TECHNIQUE</u> : Apart from the elegance of this formalism, **normal coordinates** can be a useful solving technique because they **decouple** the problem by modes.

(1) The equations of motion are $\mathbf{M}_{ki}\ddot{x}_i = -\mathbf{K}_{kj}x_j$ in x-space, with each of the ODEs involving in general *all* of the coordinates x_i. In ξ-space, the EOMs decouple to $\ddot{\xi}_m = -\omega_m^2 \xi_m$: one separated ODE for each normal coordinate ξ_m. If our system has <u>damping and/or driving forces</u> to complicate the EOMs, decoupling the EOMs may be helpful.

(2) **Initial conditions** are almost always *much* easier to deal with in ξ-space. Why? The normal coordinates decouple not only the EOMs but also their *solutions* by modes: each normal-coordinate solution is $\xi_m(t) = A_m \cos(\omega_m t - \delta_m) = \tilde{A}_m e^{i\omega_m t}$ or equivalently $B_m \cos(\omega_m t) + C_m \sin(\omega_m t)$, so it has <u>2 adjustable parameters</u> that are <u>completely independent (!!!)</u> of all the other adjustable parameters in your n-dimensional system.

Problem 1 : Normalized Basis & Normal Coordinates for Double Pendulum

Let's explore our new concepts using the double pendulum, where the {upper, lower} pendula have lengths $\{l_1, l_2\}$, attached masses $\{m_1, m_2\}$, and make angles $\{\phi_1, \phi_2\}$ with the vertical. Using ϕ_1, ϕ_2 as our generalized coordinates, the mass and spring matrices for small oscillations of the general double pendulum are:

$$\mathbf{M} = m_1 l_1^2 \begin{pmatrix} 1+\alpha & \alpha\lambda \\ \alpha\lambda & \alpha\lambda^2 \end{pmatrix} \quad \& \quad \mathbf{K} = m_1 l_1 g \begin{pmatrix} 1+\alpha & 0 \\ 0 & \alpha\lambda \end{pmatrix} \quad \text{where } \alpha \equiv \frac{m_2}{m_1} \ \& \ \lambda \equiv \frac{l_2}{l_1}$$
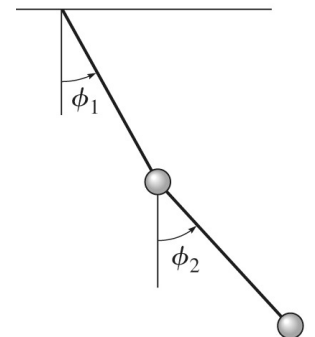
Deriving these results is great practice, but since you have already solved a triple pendulum, it's not for points.

(a) Find the normal modes (frequencies and eigenvectors) for the following double-pendulum configuration:

$$m_1 = 3, \quad m_2 = 1, \quad l_1 = l_2 = \frac{1}{2} \quad \rightarrow \quad \mathbf{M} = \frac{1}{4}\begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{K} = \frac{1}{4}\begin{pmatrix} 8g & 0 \\ 0 & 2g \end{pmatrix}$$

(b) Show explicitly that the eigenvectors $\vec{a}_S$ (slow mode) and $\vec{a}_F$ (fast mode) are not orthogonal according to the ordinary dot product, but *are* orthogonal using the new inner product we derived for normal-mode space. FYI: If you were very astute, you may have already used the orthogonality relation to *find* one of the eigenvectors; if so, bravo!
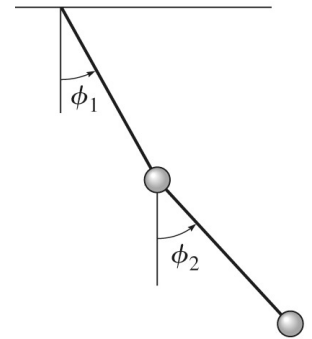
(c) Use your new skills to <u>normalize</u> the eigenvectors, i.e. to obtain $\hat{a}_S$ and $\hat{a}_F$.

We now turn to the **normal coordinates** ξ_S and ξ_F for this system. Until now, we have only used normal coordinates as a trick for solving 2-DOF systems that are symmetric under the exchange of the two coordinates, by decoupling the equations of motion. Well, a complete set $\xi_1,...\xi_n$ can be obtained for *all* linear oscillator problems, and they *always* decouple the n equations of motion. You can regard that as their definition: the ξ's are the coordinates that yield n completely decoupled EOMs. Unfortunately, it is generally not possible to guess what they are in advance, so they are only useful as a trick for *finding* the normal modes in a few simple cases. But the normal coordinates have other useful properties, so let's explore them!

(d) As we know, the general solution for our double pendulum is the superposition of the two normal modes: $\vec{\phi}(t) = \tilde{A}_S e^{i\omega_S t}\hat{a}_S + \tilde{A}_F e^{i\omega_F t}\hat{a}_F$. Using the definition $\xi_m(t) = \langle \hat{a}_m | \vec{\phi}(t) \rangle$ and your normalized eigenvectors, determine the $\xi_S(t)$ and $\xi_F(t)$ <u>as a function of time</u>. Do you see how they are the *components* of $\vec{\phi}(t)$ in our $\hat{a}_n$ basis? Do you see how each $\xi_m(t)$ gives the behaviour of a *single mode m*?

(e) Now use the definition $\xi_m = \langle \hat{a}_m | \vec{\phi} \rangle$ in a different way: instead of dropping in the full time-dependent solution $\vec{\phi}(t)$ on the right-hand side of that inner product, just drop in the coordinate vector $\vec{\phi} = (\phi_1, \phi_2)$. This time you will obtain ξ_S and ξ_F <u>as a function of your generalized coordinates</u> ϕ_1 and ϕ_2.

(f) You just found the transformation from the angle coordinates ϕ_1, ϕ_2 to the normal coordinates ξ_S, ξ_F. Do these new coordinates really give <u>decoupled EOMs</u> as advertised? Let's find out! Write down the two equations of motion in terms of angles, then transform them to normal coordinates. What new EOMs do you get? Reminder: you can read off the EOMs immediately from $\mathbf{M}$ and $\mathbf{K}$ (check lecture 1 if you've forgotten).

(g) We now have two coordinate systems, and so two ways of writing the general solution for our system:

$$\begin{pmatrix} \phi_1(t) \\ \phi_2(t) \end{pmatrix} = \tilde{A}_S e^{i\omega_S t}\begin{pmatrix} 1/\sqrt{3} \\ 2/\sqrt{3} \end{pmatrix} + \tilde{A}_F e^{i\omega_F t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \xi_S(t) \\ \xi_F(t) \end{pmatrix} = \tilde{\alpha}_S e^{i\omega_S t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \tilde{\alpha}_F e^{i\omega_F t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These are important expressions ... to study them further, demonstrate explicitly that the $\tilde{A}_{S,F}$ and $\tilde{\alpha}_{S,F}$ coefficients are EXACTLY THE SAME. Possible strategy: use (e).

(h) Normal coordinates are the best way to deal with <u>initial conditions</u>. The general solution is:

$$\begin{pmatrix} \xi_S(t) \\ \xi_F(t) \end{pmatrix} = \begin{pmatrix} \tilde{A}_S e^{i\omega_S t} \\ \tilde{A}_F e^{i\omega_F t} \end{pmatrix} = \begin{pmatrix} A_S \cos(\omega_S t - \delta_S) \\ A_F \cos(\omega_F t - \delta_S) \end{pmatrix} = \begin{pmatrix} B_S \cos(\omega_S t) + C_S \sin(\omega_S t) \\ B_F \cos(\omega_F t) + C_F \sin(\omega_F t) \end{pmatrix}$$

That last form is *ideal* for initial conditions specified at $t = 0$. Use it and part (e) to fit the B's and C's that match the following initial conditions:

at $t = 0$, $\phi_1 = \phi_2 = 2$ while $\dot{\phi}_1 = 0$ and $\dot{\phi}_2 = 1$.

That gives you $\xi_S(t)$ and $\xi_F(t)$; transform back to ϕ-space to obtain the solutions $\phi_1(t)$ and $\phi_2(t)$ that satisfy the above initial conditions. You may use the symbols $B_{S,F}$, $C_{S,F}$, and $\omega_{S,F}$ in your final answer.

Problem 2 : Unnormalized Basis Vectors

Normalizing our eigenvectors from $\vec{a}_m$ to $\hat{a}_m \equiv \vec{a}_m/|\vec{a}_m|$ makes many of the formulae in our collection elegant, specifically those that involve normal coordinates and allow us to transform between q-space and ξ-space … but honestly, it is an annoyance as it usually introduces irritating square roots to carry around. You'll be happy to learn the we can work with **unnormalized basis vectors**, as long as we change some of those formulae.

Here are the defining elements of our unnormalized IPS, with the modified formulae highlighted in **blue**:

- **Space** : $|\vec{q}(t)\rangle \equiv$ all solutions of a particular linear oscillator system
- **Inner Product** : $\langle \vec{q}_1|\vec{q}_2\rangle \equiv \vec{q}_1^T \mathbf{M}\vec{q}_2$ and associated **magnitude** : $|\vec{q}|^2 \equiv \langle \vec{q}|\vec{q}\rangle$
- **Basis** : $|\vec{a}_m\rangle$ of eigenvectors defined by $\mathbf{K}\vec{a}_m = \omega_m^2 \mathbf{M}\vec{a}_m$ with no normalization
- **Basis is Orthogonal** : $\langle \vec{a}_n|\vec{a}_m\rangle = \delta_{nm}|\vec{a}_n|^2$

We have two more sections to modify. That's your job!

(a) The next section determines how we <u>project out</u> the <u>normal coordinates</u> from a solution in q-space. Remember: each normal coordinate ξ_m is the <u>COMPONENT</u> of the solution that lies along the mode m … but now the basis vectors $\vec{a}_m$ representing these modes do *not* have magnitude 1 ...

- **Completeness** for $\vec{q}(t)$ and **Normal Coordinates** ξ_m :

    ξ_m is the **component** of $\vec{q}$ along mode m : $\quad |\vec{q}(t)\rangle = \sum_m |\vec{a}_m\rangle \dfrac{\langle \vec{a}_m|\vec{q}(t)\rangle}{?} \equiv \sum_m |\vec{a}_m\rangle \xi_m(t) = \sum_m \vec{a}_m \, \tilde{A}_m e^{i\omega_m t}$

    ξ_m is **projected out** of $\vec{q}$ by inner product: $\quad \xi_m(t) = \dfrac{\langle \vec{a}_m|\vec{q}(t)\rangle}{?} = \tilde{A}_m e^{i\omega_m t}$

You have to figure out what the question mark is.

HINT: The defining relation for the normal coordinates is $|\vec{q}(t)\rangle \equiv \sum |\vec{a}_m\rangle \xi_m(t) \rightarrow$ that is the <u>completeness relation</u> and it defines the normal coordinate ξ_m as the <u>COMPONENT</u> of the solution $\vec{q}(t)$ that lies along the mode m. You need to figure out how to project each ξ_m out of $\vec{q}(t)$ now that the basis elements $\vec{a}_m$ do not have magnitude 1. The hint: hit the completeness relation from the left with the projection operator $\langle \vec{a}_n|$.

INTUITION: Think <u>ANALOGY</u>. What we are doing is 100% equivalent to projecting a <u>3D-space vector</u> $\vec{q}$ onto an unnormalized set of basis vectors. Pick a set: $\{\vec{a}_i\} = \{2\hat{x}, 3\hat{y}, 5\hat{z}\}$ for example. What modification of the dot-product do you need to construct any vector as a linear combination of these basis vectors? i.e. What must you put in place of the question mark in $\vec{q} = \sum_i \vec{a}_i \dfrac{(\vec{a}_i \cdot \vec{q})}{?}$

(b) Next we address the transformation matrices that take us from q-space to ξ-space and back again.

• **Transformation** between q-space and ξ-space :

$$\text{vectors :} \qquad \vec{\xi} = \mathbf{R}\,\vec{q} \qquad \vec{q} = \mathbf{R}^{-1}\vec{\xi} \qquad \mathbf{R}^{-1} = \begin{pmatrix} | & | & \\ \vec{a}_1 & \vec{a}_2 & \cdots \\ | & | & \end{pmatrix} \qquad \mathbf{R} = ?$$

Nearly everything stays the same here, except of course the transformation matrix $\mathbf{R}^{-1}$ has to take us from ξ-space, where each mode is of the form $(0\ 0\ \ldots 0\ 1\ 0\ \ldots\ 0\ 0\ )$, to q-space, where are basis elements are now the unnormalized $\vec{a}_m$ eigenvectors instead of the normalized $\hat{a}_m$. But R itself has to change. The original version was

$$\mathbf{R} = \left(\mathbf{R}^{-1}\right)^T \mathbf{M}$$

(b1) First, prove that this relation is true for <u>normalized</u> basis vectors $\hat{a}_m$ by doing the following:

(i) Write the orthonormality relation $\langle \hat{a}_n | \hat{a}_m \rangle = \delta_{nm}$ in matrix form. You should get a product of *three*

matrices on the left; the convenient notation $\begin{pmatrix} | & | & \\ \hat{a}_1 & \hat{a}_2 & \cdots \\ | & | & \end{pmatrix}$ will help you to write two of them.

(ii) Spot the matrix $\mathbf{R}^{-1}$ in your expression, then use the fact that $\mathbf{R}\mathbf{R}^{-1} = \mathbf{1}$ to identify the matrix $\mathbf{R}$.

(b2) Repeat this procedure with the modified orthogonality relation and modified matrix $\mathbf{R}^{-1}$ we need for <u>unnormalized</u> basis vectors. What is $\mathbf{R}$ now?

(c) Finally, the tensor transformations:

$$\text{tensors :} \qquad \mathbf{M}^\xi = \left(\mathbf{R}^{-1}\right)^T \mathbf{M}\mathbf{R}^{-1} \qquad \rightarrow \qquad \mathbf{M}^\xi_{mn} = \delta_{mn} \ ? \quad \& \quad \mathbf{K}^\xi_{mn} = \omega_m^2 \delta_{mn} \ ?$$

You should check that the tensor transform formula $\mathbf{M}^\xi = \left(\mathbf{R}^{-1}\right)^T \mathbf{M}\mathbf{R}^{-1}$ is unchanged by going through the derivation from lecture; you will see that nothing needs to be altered. Given the changes we made to $\mathbf{R}$ and $\mathbf{R}^{-1}$, do the forms of the $\mathbf{M}^\xi$ and $\mathbf{K}^\xi$ tensors in ξ-space change? Please calculate <u>both</u> and determine what that "?" is. (You should find it is the same for $\mathbf{M}$ and $\mathbf{K}$. Also, we will do the derivation in the normalized case in the next lecture; you could wait until then if you like.)

Problem 3 : Driven 3m2s System   *Qual Problem*

Three identical blocks of mass $m = 1$ are placed in a line on a frictionless horizontal table and connected by identical springs of spring-constant $k = 1$. With the $+x$ direction pointing to the right, we number the blocks as 1,2,3 from left to right, and define x_1, x_2, and x_3 to be their x-positions relative to equilibrium. The blocks are initially at rest at $x_1 = x_2 = x_3 = 0$. At time $t = 0$, an <u>external driving force</u> $\vec{F} = f\cos(\omega t)\hat{x}$ is applied to <u>block 1</u>.

Calculate $x_3(t) = $ the motion of block 3 for times $t \geq 0$. Tactics: *do you switch to normal coordinates or not?* It is a tradeoff. Here is a little summary of what will happen if you use ξ or not:

Step	Not using ξ	Using ξ
(1) Find homogeneous solution	*usual procedure, same in both methods*	
(2) Find particular solution	easy	transformation algebra : go to ξ-space
(3) Apply initial conditions	horrible algebra	easy
(4) → final solution for $x_3(t)$	trivial	transformation algebra : return to x-space

Of course the best thing is to try *both* methods and see which you prefer. :-) Also remember problem 2: as long as you know what you're doing, you can save some algebra by *not* normalizing your basis vectors.