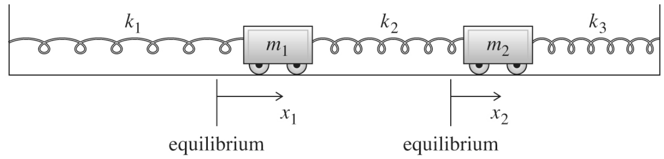
Physics 326 – Homework #3 **due Friday 1 pm**

Problem 1 : Normal Coordinates



The **normal coordinates** ξ_i of a coupled oscillator problem are the coordinates that **decouple the equations of motion**. If you could figure out what they are by just staring at your EOMs, these problems would be very simple! Unfortunately, in practice there are very few cases where you can do this. One such case is problems with two DOFs, described by the normal coordinates q_1 and q_2, where the system is **symmetric under the exchange of q_1 and q_2**. Since the EOMs are *unchanged* when you swap q_1 and q_2, here's what you do to decouple them:

- Add and subtract the EOMs to give two new EOMs.
- Identify the single linear combination of q's that appears in each of these two EOMs $\rightarrow$ these are the normal coordinates ξ_+ and ξ_-. (For 1 $\leftrightarrow$ 2 symmetric systems, they will be $\xi_+ \equiv q_1 + q_2$ and $\xi_- \equiv q_1 - q_2$.)
- Rewrite the EOMs in terms of ξ_+ and ξ_- … et voilà! You have decoupled EOMs.

(a) The figure shows our standard 2-mass-3-spring system. For this problem, we will only consider the highly symmetric case of equal masses $m_1=m_2=m$ and equal springs $k_1=k_2=k_3=k$. Write down the EOMs of this simple system in terms of x_1 and x_2 then follow the bulleted procedure above to introduce coordinates ξ_+ and ξ_- that give you decoupled EOMs. Finally, write your decoupled EOMs in matrix form, $\mathbf{M}^{\xi}\ddot{\vec{\xi}} = -\mathbf{K}^{\xi}\vec{\xi}$. What elegant property do the mass (M) and spring (K) matrices have when they are written in "ξ-space", i.e. in terms of normal coordinates?

(b) Now let's subject each mass to a linear damping force $-b\vec{v} = -2\beta m\vec{v}$ (same β for both masses). Use the method of normal coordinates to solve this problem, i.e. identify two coordinates ξ_+ and ξ_- that are linear combinations of x_1 and x_2 that decouple the equations of motion.

(c) Using damped 1D-oscillator skills, solve your decoupled equations of motion to obtain the general solutions for the normal coordinates, $\xi_+(t)$ and $\xi_-(t)$. Assume that $\beta^2 < k/m$ so that the oscillations are underdamped.

(d) Find $x_1(t)$ and $x_2(t)$ for the following initial conditions (ICs) : $x_1(0) = A$ and $x_2(0) = v_1(0) = v_2(0) = 0$.

TACTICS: Since the general solutions are so simple in ξ-space (only one mode for each normal coordinate!) ξ-space is great for applying ICs to obtain a specific solution from a general one … *if* you work efficiently. You can do one of these two things to get the specific solutions $x_1(t)$ and $x_2(t)$ that you seek :

1. Transform the *general solutions* from ξ-space to x-space, then apply the ICs to get the specific solutions.
2. Transform the *initial conditions* from x-space to ξ-space, apply them in ξ-space, then transform the *specific solutions* from ξ-space to x-space.

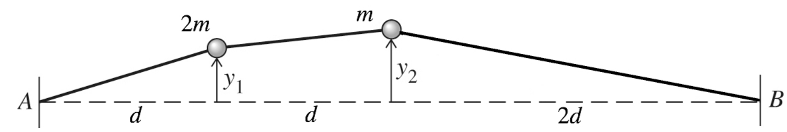
As always, try both if you can, but you will find tactic #2 to be more efficient! Also, applying initial conditions to an oscillator is usually much easier when you use the form $B\cos(\omega t) + C\sin(\omega t)$ instead of $A\cos(\omega t - \delta)$.

(e-**NOT FOR POINTS**) For your edification, re-solve the damped-oscillator system (parts b,c,d) without using normal coordinates at all, i.e. using the same technique you used for last week's damped oscillator system. Just write the equations of motion in matrix form including a damping term, $\mathbf{M}\ddot{\vec{x}} + \mathbf{D}\dot{\vec{x}} + \mathbf{K}\vec{x} = 0$, hypothesize normal-mode solution form, $\vec{x}(t) = \vec{\tilde{A}}e^{\tilde{\omega}t}$ with a complex exponent and complex amplitude, and solve for the complex eigenvalues $\tilde{\omega}$ and eigenvectors $\vec{\tilde{A}}$ using standard techniques. When you're done, ask yourself: How much easier was the normal-coordinate solution? I think you will find the answer is "not much". Normal coordinates are *conceptually* important – as we will see! – but not all that much help as a solving technique.

Problem 2 : 3 Beads and Springs on a Ring *Qual Problem*

Consider a frictionless rigid horizontal hoop of radius R. Onto this hoop we thread three beads with masses $2m$, m, and m; between the beads we thread three identical springs on the hoop, each with force constant k. Find all three normal modes, along with their frequencies, and describe each mode with a sketch.

Problem 3 : Transverse Modes



Two particles, of masses $2m$ and m, are secured to a light string of total length $4d$ that is stretched to tension T_0 between two fixed supports. As shown, the masses are not evenly spaced along the string. The masses undergo small transverse oscillations, where their transverse displacements from equilibrium, y_1 and y_2, are kept to very small values compared with the length-scale d of the string. (This system is roughly similar to the vibrations of a taut violin string.)

(a) We must first obtain a formula for the potential energy $U(\Delta y)$ of a string segment under tension when one of its ends is moved transversely by an amount Δy. Start with this **initial situation**: a string of length d lies along the z axis and is under tension T_0. The left-hand end of the string is at the origin. For convenience, define the potential energy U to be 0 when the string is in this state. Now consider this **final situation**: the left end of the string is still at the origin but the right end has been moved "sideways" to the position $(x, y, z) = (0, \Delta y, d)$. The transverse displacement Δy is very small compared to the length of the string: $\Delta y \ll d$. Calculate the potential energy $U(\Delta y)$ to **lowest non-vanishing order** in $\Delta y/d \ll 1$. Your formula must involve only the given parameters Δy, d, and T_0.

GUIDANCE: If you stretch a string by a very small amount, the change in the force=tension it exerts will be linearly proportional to the amount of stretching, simply because *linear* is the *first term* in the Taylor expansion of the force and there is no physical reason for this lowest-order term to be zero. Thus, we can treat the taut string like a stretched linear spring, exerting a force of magnitude $F = k\,(l - l_0)$ along its length. The only issue is that we aren't given a spring constant k and an unstretched length l_0 to characterize the string; instead we are given d and T_0. In brief: treat the string as a spring, just express everything in terms of d, T_0, and Δy instead of the usual parameters k, l_0 and l. Important: d is *not* the unstretched length l_0! The string is *under tension* when it has length d, so d must be *greater* than l_0.

(b) Find the normal frequencies of transverse oscillation for this system. You will find it useful throughout this problem to define the constant $\alpha \equiv T_0/(dm)$ → using it will greatly simplify your expressions!

(c) Write down the general solution for $y_1(t)$ and $y_2(t)$.

(d) Is the general motion you just calculated periodic? Explain why or why not, and if it is, give the period of the general motion.

Problem 4 : 4-Atom Ring Molecule *Qual Problem*

To study the vibrational spectrum of a ring molecule like benzene, one can reasonably approximate the molecule's atoms / sub-molecules as beads placed on a ring with springs between them. Let's try a 4-element ring molecule: consider four identical beads of mass m placed on a ring with springs of equal strength k running along the ring between the beads. Using as generalized coordinates the positions x_1, x_2, x_3, x_4 of the 4 beads measured along the ring relative to equilibrium, do the following: determine the four normal modes of the system, provide a small sketch of each mode so you can visualize it, and make sure your four modes are orthogonal to each other.

Important guidance follows so don't forget to turn the page.

ORTHOGONALITY: Two modes are **orthogonal** when their eigenvectors $\vec{A}_1$ and $\vec{A}_2$ are orthogonal, which means that the **inner product of the eigenvectors is zero**: $\langle \vec{A}_1 | \vec{A}_2 \rangle = 0$. As [was || will-be] shown in lecture, the only inner product that makes sense for the eigenvectors of a small-oscillation system is $\langle \vec{A}_1 | \vec{A}_2 \rangle \equiv \vec{A}_1^{\,T} \mathbf{M} \vec{A}_2$. This operation is proportional to the familiar dot product whenever the mass matrix **M** is proportional to the identity matrix **1**. (If that statement is not obvious to you, please ask!!!) For our 4-atom ring molecule $\mathbf{M} = m\mathbf{1}$.

☞ Thus, ***for this problem***, "two modes are orthogonal" means "the dot product of their eigenvectors is zero."

DEGENERATE MODES: You should find that two of this system's normal modes are **degenerate**, which means that two **eigenfrequencies are the same**. You must find the eigenvector for each mode; let's call them $\vec{A}_1$ and $\vec{A}_2$. As you will discover, you will not have enough conditions to completely determine both $\vec{A}_1$ & $\vec{A}_2$. Instead you must make an <u>arbitrary choice</u> when building them. Use this common tactic: set one of the free components of $\vec{A}_1$ to zero, then figure out $\vec{A}_2$ using <u>orthogonality</u>. As described in the previous paragraph, for *this* problem that means "ensure that $\vec{A}_1 \cdot \vec{A}_2 = 0$ ".

HINT: The 4x4 matrix $\mathbf{M}\omega^2 - \mathbf{K}$ can be hugely simplified by introducing a variable $\alpha \equiv \left(m\omega^2 / k - 2 \right)$.

The appendix is *not* needed for this homework, but it provides a structured summary of what we will develop in this week's lectures … might be helpful. ☺

APPENDIX : Complete Formula Set for **Inner Product Space** Description of **Normal Mode Solutions**

- **Space** : $|\vec{q}(t)\rangle \equiv$ all solutions of a particular linear oscillator system
- **Inner Product** : $\langle \vec{q}_1 | \vec{q}_2 \rangle \equiv \vec{q}_1^{\,T} \mathbf{M} \vec{q}_2$ and associated **magnitude** : $|\vec{q}|^2 \equiv \langle \vec{q} | \vec{q} \rangle$
- **Basis** : $|\hat{a}_m\rangle$ of eigenvectors defined by $\mathbf{K}\vec{a}_m = \omega_m^2 \mathbf{M}\vec{a}_m$ and normalization $\hat{a}_m \equiv \vec{a}_m / |\vec{a}_m|$
- **Basis is Orthonormal** : $\langle \hat{a}_n | \hat{a}_m \rangle = \delta_{nm}$
- **Completeness** for $\vec{q}(t)$ and **Normal Coordinates** ξ_m :

  ξ_m is the **component** of $\vec{q}$ along mode m : $|\vec{q}(t)\rangle = \sum\limits_{\text{modes } m} |\hat{a}_m\rangle\langle \hat{a}_m | \vec{q}(t)\rangle \equiv \sum\limits_m \hat{a}_m \; \xi_m(t) = \sum\limits_m \hat{a}_m \; \tilde{A}_m e^{i\omega_m t}$

  ξ_m is **projected out** of $\vec{q}$ by inner product: $\xi_m(t) = \langle \hat{a}_m | \vec{q}(t) \rangle = \tilde{A}_m e^{i\omega_m t} = A_m \cos(\omega_m t - \delta_m)$

- **Transformation** between q-space and ξ-space :

  vectors : $\vec{\xi} = \mathbf{R}\vec{q}$ $\quad \vec{q} = \mathbf{R}^{-1}\vec{\xi}$ $\quad \mathbf{R}^{-1} = \begin{pmatrix} | & | & \\ \hat{a}_1 & \hat{a}_2 & \dots \\ | & | & \end{pmatrix}$ $\quad \mathbf{R} = \left(\mathbf{R}^{-1}\right)^T \mathbf{M}$

  tensors : $\mathbf{M}^{\xi} = \left(\mathbf{R}^{-1}\right)^T \mathbf{M}\mathbf{R}^{-1} \quad \rightarrow \quad \mathbf{M}^{\xi}_{mn} = \delta_{mn} \quad \& \quad \mathbf{K}^{\xi}_{mn} = \omega_m^2 \delta_{mn}$

  inhomogeneous EOM : $\mathbf{M}\ddot{\vec{x}} + \mathbf{K}\vec{x} = \vec{F}$ in q-space $\quad \rightarrow \quad \mathbf{M}^{\xi}\ddot{\vec{\xi}} + \mathbf{K}^{\xi}\vec{\xi} = \left(\mathbf{R}^{-1}\right)^T \vec{F}$ in ξ-space