Derivation of $\vec{L}^{(B)} = \mathbf{I}^{(B)}\vec{\omega}$

$$\vec{L}^{(B)} = \int \vec{r}^{(B)} \times dm\, \dot{\vec{r}}^{(B)}$$

$\vec{r}^{(B)}$ points from body-fixed ref. point (B) to each piece of mass dm, so

$\vec{r}^{(B)}$ is body-fixed $\rightarrow$ $\dot{\vec{r}}^{(B)} = \vec{\omega} \times \vec{r}^{(B)}$

$$\begin{aligned}\vec{L}^{(B)} &= \int dm\, \vec{r}^{(B)} \times (\vec{\omega} \times \vec{r}^{(B)}) \\ &= \int dm\, \vec{\omega}\left(\vec{r}^{(B)} \cdot \vec{r}^{(B)}\right) - \vec{r}^{(B)}\left(\vec{r}^{(B)} \cdot \vec{\omega}\right) \\ &= \int dm\, \vec{\omega}\left|r^{(B)}\right|^2 - \vec{r}^{(B)} \sum_j r_j^{(B)} \omega_j\end{aligned}$$

$$\begin{aligned}L_i^{(B)} &= \int dm\, \omega_i \left|r^{(B)}\right|^2 - r_i^{(B)} \sum_j r_j^{(B)} \omega_j \\ &= \sum_j \int dm\, \omega_j \delta_{ij} \left|r^{(B)}\right|^2 - r_i^{(B)} r_j^{(B)} \omega_j \\ &= \sum_j I_{ij}^{(B)} \omega_j \quad \text{with} \quad \boxed{I_{ij}^{(B)} = \int dm\, \delta_{ij} \left|r^{(B)}\right|^2 - r_i^{(B)} r_j^{(B)}}\end{aligned}$$

$$\mathbf{I} = \int dm \begin{pmatrix} y^2+z^2 & -xy & -xz \\ \cdot & z^2+x^2 & -yz \\ \cdot & \cdot & x^2+y^2 \end{pmatrix}$$

with implicit (B) ref. pt. on everything

Derivation of Euler's Equations

$\vec{\tau}^{(A)} = \dfrac{d\vec{L}^{(A)}}{dt}$ expressed in **Body Frame** S* = $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$

$$\vec{\tau} = \frac{d}{dt}\left(\mathbf{I}\vec{\omega}\right) = \frac{d}{dt}\left(I_1\omega_1\hat{e}_1 + I_2\omega_2\hat{e}_2 + I_3\omega_3\hat{e}_3\right)$$

Two sources of time-dep: (1) ω_i and (2) $\hat{e}_i$

$$\begin{aligned}\vec{\tau} &= I_1\dot{\omega}_1\hat{e}_1 + I_2\dot{\omega}_2\hat{e}_2 + I_3\dot{\omega}_3\hat{e}_3 \\ &\quad + I_1\omega_1\left(\vec{\omega} \times \hat{e}_1\right) + I_2\omega_2\left(\vec{\omega} \times \hat{e}_2\right) + I_3\omega_3\left(\vec{\omega} \times \hat{e}_3\right)\end{aligned}$$

$$\begin{aligned}\vec{\omega} \times \hat{e}_1 &= \left(\omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3\right) \times \hat{e}_1 = 0 - \omega_2\hat{e}_3 + \omega_3\hat{e}_2 \\ \vec{\omega} \times \hat{e}_2 &= \left(\omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3\right) \times \hat{e}_2 = +\omega_1\hat{e}_3 + 0 - \omega_3\hat{e}_1 \\ \vec{\omega} \times \hat{e}_3 &= \left(\omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3\right) \times \hat{e}_3 = -\omega_1\hat{e}_2 + \omega_2\hat{e}_1 + 0\end{aligned}$$

$$\boxed{\begin{aligned}\tau_1 &= I_1\dot{\omega}_1 + \omega_2\omega_3\left(I_3 - I_2\right) \\ \tau_2 &= I_2\dot{\omega}_2 + \omega_3\omega_1\left(I_1 - I_3\right) \\ \tau_3 &= I_3\dot{\omega}_3 + \omega_1\omega_2\left(I_2 - I_1\right)\end{aligned}}$$

cycle $1 \rightarrow 2 \rightarrow 3$