Gauss-Green-Stokes (GGS) & Related Theorems

general form: $\int_{\mathbb{R}} df = \oint_{\partial\mathbb{R}} f$

Grad $\int_{\vec{a}}^{\vec{b}} \vec{\nabla} V \cdot d\vec{l} = V(\vec{b}) - V(\vec{a})$

Stokes $\int_{Surface} (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} = \oint_{\partial Surface} \vec{E} \cdot d\vec{l}$

Gauss $\int_{Volume} (\vec{\nabla} \cdot \vec{E})\, dV = \oint_{\partial Volume} \vec{E} \cdot d\vec{A}$

Irrotational Fields

1. $\vec{\nabla} \times \vec{F} = 0$ everywhere
2. $\oint \vec{F} \cdot d\vec{l} = 0 \;\forall$ closed loop
3. $\int_{\vec{a}}^{\vec{b}} \vec{F} \cdot d\vec{l}$ is path-independent
4. $\vec{F} = \vec{\nabla} g$ for some $g(\vec{r})$

Divergenceless Fields

1. $\vec{\nabla} \cdot \vec{B} = 0$ everywhere
2. $\oint \vec{B} \cdot d\vec{A} = 0 \;\forall$ closed surface
3. $\int_S \vec{B} \cdot d\vec{A}$ depends only on ∂S
4. $\vec{B} = \vec{\nabla} \times \vec{A}$ for some $\vec{A}(\vec{r})$

Vector-Calculus Identities

Triple Products: $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$ $\qquad \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

1st Deriv:

$\vec{\nabla}(fg) = f(\vec{\nabla} g) + g(\vec{\nabla} f)$

$\vec{\nabla} \cdot (f\vec{A}) = f(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot (\vec{\nabla} f)$

$\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) - \vec{A} \times \vec{\nabla} f$

$\vec{\nabla}(\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{A}$

$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$

2nd Deriv: $\vec{\nabla} \times (\vec{\nabla} f) = 0$ $\qquad \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$ $\qquad \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

Cartesian Coordinates

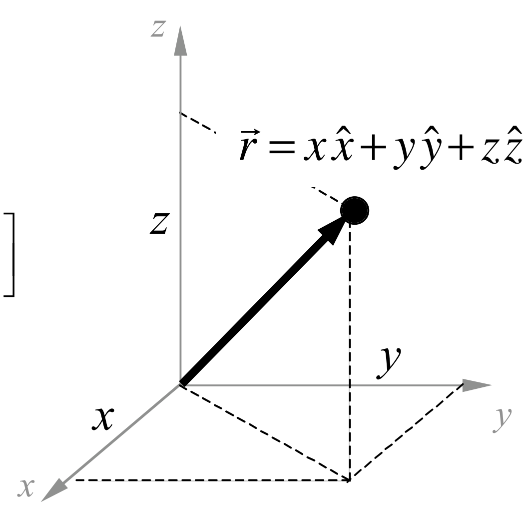
Line Element: $d\vec{l} = dx\ \hat{x} + dy\ \hat{y} + dz\ \hat{z}$

Gradient: $\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}$

Divergence: $\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$

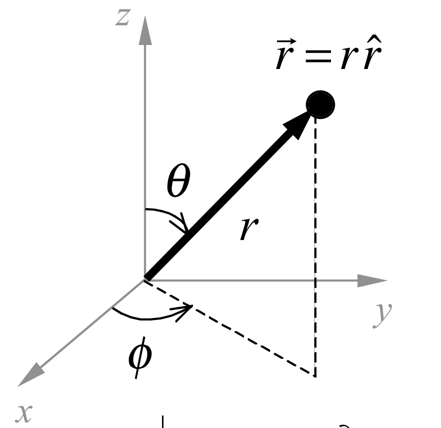
Curl: $\vec{\nabla} \times \vec{E} = \hat{x}\left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right] + \hat{y}\left[\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right] + \hat{z}\left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right]$

Laplacian: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$



Spherical Coordinates

Line Element: $d\vec{l} = dr\ \hat{r} + r\,d\theta\ \hat{\theta} + r\sin\theta\,d\phi\ \hat{\phi}$



$$x = r\sin\theta\cos\phi \qquad \hat{x} = \sin\theta\cos\phi\ \hat{r} + \cos\theta\cos\phi\ \hat{\theta} - \sin\phi\ \hat{\phi}$$
$$y = r\sin\theta\sin\phi \qquad \hat{y} = \sin\theta\sin\phi\ \hat{r} + \cos\theta\sin\phi\ \hat{\theta} + \cos\phi\ \hat{\phi}$$
$$z = r\cos\theta \qquad \hat{z} = \cos\theta\ \hat{r} - \sin\theta\ \hat{\theta}$$

$$r = \sqrt{x^2+y^2+z^2} \qquad \hat{r} = \sin\theta\cos\phi\ \hat{x} + \sin\theta\sin\phi\ \hat{y} + \cos\theta\ \hat{z}$$
$$\theta = \tan^{-1}(\sqrt{x^2+y^2}\ /\ z) \qquad \hat{\theta} = \cos\theta\cos\phi\ \hat{x} + \cos\theta\sin\phi\ \hat{y} - \sin\theta\ \hat{z}$$
$$\phi = \tan^{-1}(y/x) \qquad \hat{\phi} = -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}$$

	∂_r	∂_θ	∂_ϕ
$\hat{r}$	0	$\hat{\theta}$	$\sin\theta\hat{\phi}$
$\hat{\theta}$	0	$-\hat{r}$	$\cos\theta\hat{\phi}$
$\hat{\phi}$	0	0	$-\sin\theta\hat{r} - \cos\theta\hat{\theta}$

Gradient: $$\vec{\nabla}V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\phi}$$

Laplacian: $$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}$$

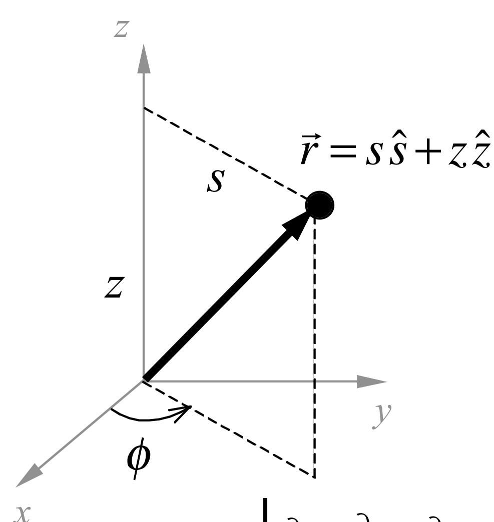
Divergence: $$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ E_\theta) + \frac{1}{r\sin\theta}\frac{\partial E_\phi}{\partial\phi}$$

Curl: $$\vec{\nabla}\times\vec{E} = \frac{\hat{r}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\ E_\phi) - \frac{\partial E_\theta}{\partial\phi}\right] + \frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi} - \frac{\partial}{\partial r}(rE_\phi)\right] + \frac{\hat{\phi}}{r}\left[\frac{\partial}{\partial r}(rE_\theta) - \frac{\partial E_r}{\partial\theta}\right]$$

Acceleration: $$\vec{a} = \hat{r}\left[\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta\right] + \hat{\theta}\left[r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right] + \hat{\phi}\left[\sin\theta(r\ddot{\phi} + 2\dot{r}\dot{\phi}) + \cos\theta(2r\dot{\theta}\dot{\phi})\right]$$

Cylindrical Coordinates

Line Element: $d\vec{l} = ds\ \hat{s} + s\,d\phi\ \hat{\phi} + dz\ \hat{z}$



$$x = s\cos\phi \qquad \hat{x} = \cos\phi\ \hat{s} - \sin\phi\ \hat{\phi}$$
$$y = s\sin\phi \qquad \hat{y} = \sin\phi\ \hat{s} + \cos\phi\ \hat{\phi}$$
$$z = z \qquad \hat{z} = \hat{z}$$

$$s = \sqrt{x^2+y^2} \qquad \hat{s} = +\cos\phi\ \hat{x} + \sin\phi\ \hat{y}$$
$$\phi = \tan^{-1}(y/x) \qquad \hat{\phi} = -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}$$
$$z = z \qquad \hat{z} = \hat{z}$$

	∂_s	∂_ϕ	∂_z
$\hat{s}$	0	$\hat{\phi}$	0
$\hat{\phi}$	0	$-\hat{s}$	0
$\hat{z}$	0	0	0

Gradient: $$\vec{\nabla}V = \frac{\partial V}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial V}{\partial\phi}\hat{\phi} + \frac{\partial V}{\partial z}\hat{z}$$

Laplacian: $$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial\phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Divergence: $$\vec{\nabla}\cdot\vec{E} = \frac{1}{s}\frac{\partial}{\partial s}(sE_s) + \frac{1}{s}\frac{\partial E_\phi}{\partial\phi} + \frac{\partial E_z}{\partial z}$$

Curl: $$\vec{\nabla}\times\vec{E} = \hat{s}\left[\frac{1}{s}\frac{\partial E_z}{\partial\phi} - \frac{\partial E_\phi}{\partial z}\right] + \hat{\phi}\left[\frac{\partial E_s}{\partial z} - \frac{\partial E_z}{\partial s}\right] + \frac{\hat{z}}{s}\left[\frac{\partial}{\partial s}(sE_\phi) - \frac{\partial E_s}{\partial\phi}\right]$$

Acceleration: $$\vec{a} = \hat{s}\left[\ddot{s} - s\dot{\phi}^2\right] + \hat{\phi}\left[s\ddot{\phi} + 2\dot{s}\dot{\phi}\right] + \hat{z}\left[\ddot{z}\right]$$