Problem 1 (15 points) A particle of mass $m$ moves in one-dimension while suffering a drag force opposed to its motion of magnitude $Cv^{2/3}$ where $C$ is a positive constant. It starts at the origin $x = 0$ at time $0$ with velocity $v_o > 0$, that then diminishes due to the drag.

a) What is its speed after time $t$?

b) At what time does it stop?

*c) How far does it go before stopping?

$$mdv/dt = -Cv^{2/3}; \quad \text{so} \quad \frac{dv}{v^{2/3}} = -\frac{C}{m} dt; \quad \text{so} \quad 3v^{1/3} \bigg|_{v_0}^{v} = -Ct/m$$

so $v^{1/3} - v_0^{1/3} = -Ct/3m$; so $v = v_0[1 - Ct/3mv_0^{1/3}]^3$

So $t_{stop} = 3mv_0^{1/3}/C$

$$x = v_0 \int_0^{tstop} [1 - Ct/3mv_0^{1/3}]^3 dt = \frac{v_0}{4C} \frac{3m}{4C}$$

Problem 2: (15 points) A particle of mass $m=2\text{kg}$ moves in 1-d under a force distribution with potential energy (in Joules) given by the plot (tic marks indicate Joules and meters)

Describe the possible motions of the mass qualitatively, and estimate the $x$-value of any turning points(s), if...

a) It has energy $E = 4 \text{ Joules}$

answer: It passes through the region without any turning.

b) It has energy $E = 1 \text{ Joule}$

answer: there is a turning point at about $x=4.5$. If the particle is initially going rightwards, it turns at $x = 4.5$ and goes left. If initially going left it just keeps on going, faster and slower in places, but without turning.

c) It has energy $E = -3 \text{ Joule}$

answer: it is trapped in the well, oscillating periodically between turning points at approximately $x = 3.5$ and $x = -3.5$

*d) Now assume it starts at position $x_0 = 0$ with $v_o = 2\text{m/sec}$ to the right. What is its kinetic energy there? What is its Potential energy there?

$KE = (1/2)m v_o^2 = 4 J$; $PE = (\text{from reading the plot}) -5 J$. So total energy is $-1 J$.

What is its behavior and speed as $t$ goes to $\infty$?

It ends up ultimately going leftwards (after bouncing at about $x = 4$) to $x \to -\infty$, where $PE = -2$, so $KE = 1J$ (because total $E = -1J$) so speed = $1\text{m/sec}$ (leftwards) from $1 J = (1/2) m v^2$
**Problem 3: (30 points) Consider a spaceship of small mass \( \mu \) in an elliptical orbit around the sun. The orbit has eccentricity \( e = 1/2 \) and closest approach (perihelion) a distance \( R_{\text{min}} \) from the sun. All answers are to be given in terms of \( \mu \), and the sun’s mass \( M \), and \( G \), \( \varepsilon \), and the given \( R_{\text{min}} \).

a) What is its furthest distance (aphelion) \( R_{\text{max}} \) from the sun?

We use \( r(\phi) = \frac{\alpha}{1 + \varepsilon \cos(\phi)} \); \( \alpha = \ell^2 / GM \); \( \varepsilon = \sqrt{1 + \frac{2e\ell^2}{G^2M^2}} \)

This implies \( R_{\text{min}} = \frac{\alpha}{(1+\varepsilon)} \); hence \( R_{\text{max}} = \frac{\alpha}{(1-\varepsilon)} = R_{\text{min}} \frac{1+\varepsilon}{1-\varepsilon} = 3R_{\text{min}} \)

b) What is its speed \( v_{\text{perihelion}} \) at closest approach?

We see that \( \alpha = R_{\text{min}} (1+\varepsilon) \); so \( \ell = \sqrt{\alpha GM} = \sqrt{GMR_{\text{min}}(1+\varepsilon)} = R_{\text{min}} v_{\text{perihelion}} \)

So \( v_{\text{perihelion}} = \sqrt{GM(1+\varepsilon) / R_{\text{min}}} = \sqrt{3GM / 2R_{\text{min}}} \).

We note in passing that this is slightly greater than it would be if the orbit were circular.

c) What is its speed \( v_{\text{aphelion}} \) at \( R_{\text{max}} \)?

answer: at aphelion the speed must be less than at perihelion by the ratio of the \( R \) (by angular momentum conservation) so \( v_{\text{aphelion}} = v_{\text{perihelion}} (R_{\text{min}} / R_{\text{max}}) = (1 / 3) \sqrt{3GM / 2R_{\text{min}}} \)

d) What is the magnitude \( |L| \) of its angular momentum?

\( |L| = \mu \ell = \mu \sqrt{GM(1+\varepsilon)} = \mu \sqrt{3GM_{\text{min}} / 2} \)

e) What is its energy \( E \)?

Use \( \varepsilon = \sqrt{1 + \frac{2e\ell^2}{G^2M^2}} \) to solve for \( e \) and set \( E = e\mu \).

*f) What instantaneous increase of speed \( \Delta v \) is required if the spaceship is to boost into an escape trajectory from perihelion \( r = R_{\text{min}} \)?

Escape velocity from a distance \( R \) is \( V_{\text{escape}} = (2GM/R)^{1/2} \). The boost needed is therefore

\[ \Delta v = V_{\text{escape}} - v_{\text{perihelion}} = (2GM/R_{\text{min}})^{1/2} - \sqrt{GM(1+\varepsilon) / R_{\text{min}}} = \sqrt{GM / R_{\text{min}}} \left[ \sqrt{2} - \sqrt{3/2} \right] \]

Problem 4: (20 points) A rocket of initial mass \( M_0 \) burns fuel at a constant rate \( \kappa \) (kg per second). It starts at time \( t=0 \) with zero speed and moves vertically. You may neglect drag forces but not the (constant) gravitational acceleration \( g \). Exhaust velocity relative to the rocket is \( u \).

Find its velocity \( v \) as a function of time.

We use the rocket equation adapted from the provided formulas:

\[ M(t) \dot{v} + u \dot{M} = F_{\text{external}} = -Mg \]

Thus, with \( M = M_0 - \kappa t \),
Which can be integrated:

\[ v = -u \log \left( \frac{M_o - \kappa t}{M_o} \right) - gt \]

We confirm that this gives the right answer when \( \kappa = 0 \) ( \( v = gt \), i.e. just falling)

We also confirm that it satisfies initial conditions, \( v=0 \) at \( t = 0 \)

5) Tides over a disk

The potential over a thin disk was worked out in class to be

\[ \Phi = -2\pi G \rho h \left[ \sqrt{R^2 + z^2} - z \right] \]

Two particles are placed on the z-axis, one at \( z \), another at \( z + \varepsilon \) where \( \varepsilon \) is small and positive.

What is the acceleration of the lower particle?

Find the relative acceleration of the two particles. Does the upper one fall faster or slower? By how much?

\[ \Phi = -2\pi G \rho h \left[ \sqrt{R^2 + z^2} - z \right] \]

implies \( g = -\frac{\partial z}{\partial \Phi} = 2\pi G \rho h \left[ \frac{z}{\sqrt{R^2 + z^2}} - 1 \right] \)

Notice that \( \frac{z}{\sqrt{R^2 + z^2}} < 1 \) So \( g \) is always negative; accelerations are downwards.

The quantity \( \frac{z}{\sqrt{R^2 + z^2}} \) increases with height \( z \), from zero to 1 as \( z \) goes from 0 to \( \infty \). So \( g \) is gaining positive contributions with height, so the magnitude of \( g \) is diminishing with height. The higher particle is falling less quickly.

\[ g = 2\pi G \rho h \left[ \frac{z}{\sqrt{R^2 + z^2}} - 1 \right] \]

implies

\[ g' = 2\pi G \rho h \left( \frac{1}{\sqrt{R^2 + z^2}} - \frac{z^2}{(R^2 + z^2)^{3/2}} \right) = 2\pi G \rho h \frac{R^2}{(R^2 + z^2)^{3/2}} \]

The relative acceleration is therefore \( g' \varepsilon = 2\pi G \rho h \frac{R^2}{(R^2 + z^2)^{3/2}} \varepsilon \)
6) Find the gravitational potential $\Phi$ a distance 'a' from the end of a uniform rod of length $L$ and mass $M$.

\[ \Phi = \int_0^a \frac{dM}{s} \]

With $s = a + x$, and $x$ (the distance from the right hand end into the material) runs from 0 to $L$, and $dM = dx/L$

\[ \Phi = -G \frac{M}{L} \int_0^L \frac{dx}{a + x} = -G \frac{M}{L} \log \left( \frac{a + L}{a} \right) \]

At $a >> L$ we may write $\log \left( 1 + L/a \right) \sim L/a$, so

$\Phi \approx -GM/a$