Solution to these many special cases: (a-e)

We are seeking the motion $\mathbf{r}(t)$ subsequent to specified initial conditions, and given some prescription for how $\mathbf{F}$ varies with position and velocity and time. The process involves the solution of an ODE. The ODE is second order, so we'll need two (vector) constants of integration.

Case a) $\ddot{F} = 0$:

We solve by doing two indefinite integrations, getting two constants of integration $\ddot{v}_o$ and $\ddot{r}_o$:

$$\ddot{a} = 0 \quad \Rightarrow \quad d\ddot{v} / dt = 0 \quad \Rightarrow \quad \ddot{v} = \text{constant} = \ddot{v}_o$$

$$\ddot{r}(t) = \int \ddot{v}(t) dt = \ddot{v}_o t + \ddot{r}_o$$

The constants of integration are recognized as equal to the velocity and position at time zero.

Case b) $\ddot{F} = \text{constant} = \ddot{F}_o$ (for which case (a) was a special case)

$$\ddot{a} = \ddot{F}_o / m \quad \Rightarrow$$

Now integrate each side (indefinite integral) with respect to time:

$$\ddot{v}(t) = \int (\ddot{F}_o / m) dt = \ddot{F}_o t / m + \dddot{C}_1$$

The constant of integration $\dddot{C}_1$ may be recognized as the velocity $\ddot{v}_o$ it has at time $t = 0$. If you knew instead its velocity $\ddot{v}_i$ at some other time, $t_i$, you would identify the constant of integration by solving $\ddot{v}(t_i) = \ddot{F}_o t_i / m + \dddot{C}_1$. You'd get $\dddot{C}_1 = \ddot{v}_i - \ddot{F}_o t_i / m$.

Now integrate again (indefinitely) with respect to time

$$\ddot{r}(t) = \int (\dddot{F}_o t / m + \dddot{C}_1) dt = \dddot{F}_o t^2 / 2m + \dddot{C}_1 t + \dddot{C}_2$$

The new constant of integration $\dddot{C}_2$ may be identified with the position $\dddot{r}_o$ at time zero. If instead you knew the position at some other time, you could use that to solve for $\dddot{C}_2$.

This case should be familiar from Phys 211 where you so often saw the formula

$$x = (a/2)t^2 + v_0 t + x_0$$

Do keep in mind that this familiar simple formula applies only to the one-dimensional case with $\mathbf{F} = \text{constant}$ and initial conditions specified at time zero.

case(b) continued. You could alternatively choose to do the integrals as definite integrals. Just make sure that the limits on the integrals on the right and the left side of the equation correspond. For example, if you start with
\( \ddot{a} = d\ddot{v} / dt = \ddot{F}_o / m \)  

Slap on an integral sign and a factor of dt on both sides to get
\[
\int_{t_1}^{t_2} d\ddot{v} = (\ddot{F}_o / m) \int_{t_1}^{t_2} dt
\]

(NB note the limits!)

with the understanding the times t1 and t2 correspond respectively to the velocities v1 and v2.

The result is

\[ \dddot{v}_2 - \dddot{v}_1 = (\dddot{F}_o / m) \{t_2 - t_1\} \]

There are now no constants of integration to consider, and initial conditions (at time \( t_1 \)) enter naturally into the expressions.

We now replace v2 and t2 with more generic v and t to get:

\[ \dddot{v} = (\dddot{F}_o / m) \{t - t_1\} + \dddot{v}_1 \]

and integrate again with respect to time, again using corresponding limits on the two sides
\[
\int_{t_1}^{t} \dddot{v} dt = \dddot{r}(t) - \dddot{r}(t_1) = \int_{t_1}^{t} ((\dddot{F}_o / m) \{t - t_1\} + \dddot{v}_1) dt
\]

which after evaluating will give position as a function of time in terms of the velocity \( v_1 \) at time \( t_1 \) and the position \( r_1 \) at time \( t_1 \). Again no constants of integration appear - because the integrals were definite.

Case c) \( \dddot{F} = \dddot{f}(t) \)

We need merely integrate twice with respect to time. (we'll use indefinite integrals and constants of integration here but we could do it with definite integrals if we preferred) Cases a and b are special cases of (c).

\[
\dddot{a}(t) = \dddot{f}(t) / m \quad \Rightarrow
\]

\[
d\dddot{v} = (\dddot{f}(t) / m) dt
\]

\[
\int d\dddot{v} = \int (\dddot{f}(t) / m) dt
\]

\[
\dddot{v}(t) = \dddot{F}(t) / m + \dddot{C}_1
\]

\[
\int d\dddot{r} = \int ((\dddot{F}(t) / m + \dddot{C}_1) dt
\]

\[
\dddot{r}(t) = \dddot{F}(t) / m + \dddot{C}_1 t + \dddot{C}_2
\]

where \( F(t) \) is the anti-derivative of \( f(t) \) and \( F(t) \) is the anti-derivative of \( F(t) \).

The vector constants of integration \( C_1 \) and \( C_2 \) may be solved for by invoking additional knowledge, for example the velocity and position at some particular time. \( \text{Nota Bene: The values of } C_1 \text{ and } C_2 \text{ are related to the initial velocity and positions, but are not always equal to them!} \) (See Miniquiz 2)

Case d) \( \dddot{F} = \dddot{F}(\dddot{r}) \)

The force depends on where the particle is. Cases a and b can also be considered special cases of this. We will for now restrict to 1-dimension (x).
In 1-d it takes the form: \( m \frac{dv}{dt} = F(x) \), where \( v = \frac{dx}{dt} \).

NB! You cannot profit by slapping an integral sign on both sides of \( m \frac{dv}{dt} = F(x) \) and integrating with respect to \( t \). . . . . because the right hand side isn't (yet) known as a function of time. 
\[
\int m \left( \frac{dv}{dt} \right) \, dt = mv + C = \int F(x) \, dt .
\]
This expression is correct, and you can evaluate the left side, but you cannot evaluate the right side (unless you already know \( x(t) \)), so it isn't good for much.

Trick: Multiply by \( v \):
\[
mv \frac{dv}{dt} = v F(x) = F(x) \frac{dx}{dt}
\]
now multiply by \( dt \):
\[
mv \, dv = F(x) \, dx
\]
This can be integrated:
\[
\int mv \, dv = \int F(x) \, dx
\]
If we do a definite integral, from \( v_0 \) to \( v \) on the left and \( x_0 \) to \( x \) on the right, (with the understanding that \( v \) is the speed it has when it is at \( x_0 \), and \( v \) is the speed it has when it is at \( x \) ) it becomes
\[
\frac{m}{2} [v^2 - v_0^2] = \int_{x_0}^{x} F(x) \, dx
\]
or,
\[
\frac{mv^2}{2} = \frac{mv_0^2}{2} + \int_{x_0}^{x} F(x) \, dx
\]
which gives us speed \( v \) as a function of position \( x \). This is a first integral of the second order equation of motion: \( m \frac{dx}{dt^2} = F(x) \). The next step would involve using the above to express \( v = \frac{dx}{dt} \) as a function of \( x \), and then solving the resulting differential equation for \( x(t) \). But before doing that it is instructive to re-write the above in more physical terms:

We define the kinetic energy as \( T \equiv \frac{mv^2}{2} \)
and notice that the Kinetic energy when it was at \( x_0 \) was \( T_0 \equiv \frac{mv_0^2}{2} \)

Then the above first integral of the differential equation becomes the work-energy theorem
\[
T = T_0 + \int_{x_0}^{x} F(x) \, dx = T_0 + \text{work done by } F \text{ over the interval from } x_0 \text{ to } x
\]
If we define the potential energy \( U(x) \) as the anti-derivative of \( -F(x) \), then
\[
dU/dx = -F(x) \quad \text{(the function } U(x)\text{, we note, is unique only up to some additive constant )}
\]
then the above first integral of the differential equation becomes

\[ T = T_o - U(x) + U(x_o) \quad \text{or} \quad T + U(x) = T_o + U(x_o) \]

which is (after defining total energy as \( E = T + U \)) conservation of total energy:

\[ E = T + U(x) = E_o = T_o + U(x_o) \]

---------------

The expressions derived for case (d) are essentially giving us \( v \) as a function of \( x \). We may be able to do another integral so as to determine \( x \) as a function of time. Solve \( E = T + U(x) \) for the velocity: (recalling that we know the constant \( E = E_0 \) in terms of the initial velocity and position)

Then

\[ v = \frac{dx}{dt} = \pm \sqrt{\frac{2}{m}(E - U(x))} \]

We note that \( E - U(x) \) is merely the kinetic energy at \( x \)

If you slapped an integral sign on the above and integrated with respect to \( t \),

\[ x = \int v \, dt = \pm \int \sqrt{\frac{2}{m}(E - U(x))} \, dt \]

you could do the integral on the left hand side, but not the integral on the right side.

So the above is true, but of no utility.

Instead, let us solve \( \frac{dx}{dt} = \pm \sqrt{2(E - U(x))/m} \) for \( dt \) and integrate (again being careful that the lower and upper limits correspond)

\[ \int_{t_o}^{t} dt = t - t_o = \pm \int_{x_o}^{x} \frac{dx}{\sqrt{2(E - U(x))/m}} \]

If this integral can be done analytically (possible for some functions \( U(x) \)), we'll get time as a function of \( x \). Not exactly what we wanted, but close.

Often this integral cannot be done analytically, but if not, we may not need to be reduced to the indignity of numerics. We can get a good qualitative idea as to the behavior of \( x(t) \): Consider an arbitrary \( U(x) \) (this is fig 2-14 from TM) on the next page.

We will consider various possibilities for the constant of integration, total energy \( E \) (which is known by referring to initial conditions).

* If \( E < E_0 \)
This cannot be. E can’t be less than the lowest point of the curve (labeled \(E_0\)) because if it were, the Kinetic energy \(T = E - U(x)\) would have to be negative, regardless of where the particle might be.

* If \(E = E_0\),
  the particle must be at rest at \(x_0\), where it has kinetic energy \(T = 0\). If \(E = E_0\), \(x_0\) is the only place where \(T\) is not less than zero. And of course \(T\) can’t be negative, because it is \((m/2)\) times the square of speed.

* If \(E = E_1\),
  A particle with energy \(E_1\) can only move in the interval between \(x_a\) and \(x_b\). It oscillates periodically between the turning points \(x_a\) and \(x_b\). (Please convince yourself that it is obvious why the motion must be periodic in time.) At all other values of \(x\), \(T\) would have to be negative. \(x_a\) and \(x_0\) are the Turning points, where Kinetic Energy \(T\) goes to zero, i.e, where Total Energy = \(U\). Clearly the kinetic energy at \(x_a\) or \(x_b\) is zero, so the speed will be zero there – but how do we know that it doesn’t just stay there? Answer: \(F = -U'(x) \neq 0\) so the particle has no speed there, but it is accelerating.

* if \(E = E_2\),
  A particle with energy \(E_2\) could be oscillating back and forth between turning points \(x_c\) and \(x_d\) or between turning points \(x_e\) and \(x_f\). It cannot move between these two regions (unless you are doing Quantum mechanics)

* if \(E = E_3\)
  A particle with energy \(E_3\) must have \(x \geq x_g\). It comes in from infinity on the right, turns at \(x_g\), and goes out to infinity on the right. Clearly the kinetic energy at \(x_g\) is zero, so the speed is zero there – but how do we know that it doesn’t just stay there? Answer: \(F = -U'(x_g) \neq 0\) so the particle may have no speed there, but it is accelerating towards the right.

* if \(E = E_4\)
  A particle with energy \(E_4\) can be at any position. As it passes through the pictured region it speeds up and slows down, but never turns around.
Thus the qualitative motion is deducible from the above plot of \( U(x) \). We do not have to solve the differential equation completely to know how the particle moves. The possible motions are determinable, at least qualitatively, merely by inspection of the curve \( U(x) \). Thus you may not have to do the integral 
\[
\int dx / \sqrt{2(E - U(x))/m}
\]
If you are content with such qualitative descriptions of how the particle moves (and we often are) then we are done.

If you want to know precisely where it is \( x(t) \) at any given time, you will need to do the integral
\[
\int_{t_0}^{t} dt = t - t_0 = \pm \int_{x_0}^{x} \frac{dx}{\sqrt{2(E - U(x))/m}}
\]
(This gives \( t(x) \) which you may then need to invert.)

In many cases the integral \( \int dx / \sqrt{2(E - U(x))/m} \) cannot be done analytically. There are a few in which it can be done. One such is that of HW1B.2. Another is the harmonic oscillator.

There are other qualitative statements that can be made as well. One can analyze motion near the equilibrium points and decide whether that equilibrium is stable or not. The above \( U(x) \) has five equilibrium points, at which the force \( F = -dU/dx = 0 \). The maxima (the points marked @, # and $) are all unstable. Imagine for example that the particle is at @, and is given a small nudge to the right or the left. The force is then non zero, and directed away from @, so the particle then accelerates away from the equilibrium. The minima, \( x_0 \) and the point marked *, are stable. A nudge in one direction will invoke a restoring force \( F = -dU/dx \) that pushes it back to the equilibrium. Equilibrium points have \( U'' = 0 \). They are unstable if \( U'' < 0 \), i.e a maximum. They are stable if \( U'' > 0 \), i.e at the bottom of a well.

**Simple Harmonic Oscillator – Behavior near a stable equilibrium**

As described above, it is possible to qualitatively understand the solution to the ODE
\[
m \frac{d^2 x}{dt^2} = F(x).
\]
It is possible to get a more detailed analytic understanding if \( x \) is, and remains, *in the vicinity* of one of the stable equilibrium points like \( x_0 \) or *.

Let us shift our \( x \)-coordinate so that the equilibrium position (for example \( x_0 \) in the picture) is at zero. Then

\[
m \frac{d^2 x}{dt^2} = F(x) =>
\]
\[
m \frac{d^2 x}{dt^2} = F(x(t)) = F(0) + x(t) \left( \frac{dF}{dx} \big|_0 \right) + \frac{1}{2} x(t)^2 \left( \frac{d^2 F}{dx^2} \big|_0 \right) + ...
\]
in which \( F \) has been expanded in a Taylor series. We note that \( F(0) = 0 \), by hypothesis. We neglect the term in \( x^2 \) as small (\( x \) now measures the distance from equilibrium and we are assuming that is small, so \( x^2 \) is especially small.) This leaves only the term linear in \( x \).
\[ m \frac{d^2x}{dt^2} = x(t) \left\{ \frac{dF}{dx}|_0 \right\} - x(t) \left\{ \frac{d^2U}{dx^2}|_0 \right\} = -kx(t) \]

The constant \( k \) is the curvature of \( U \) near the minimum. \( k = U''(x_{eq}) \) is positive if \( U \) is a minimum, \( U'' > 0 \), that is, if we are analyzing near a stable point. (The analysis below can be invoked near a maximum also, where the motion is unstable, in which case \( k \) is a negative number. But for now we'll assume \( k \) is positive.)

We could also expand \( U \) in the vicinity of the equilibrium, \( U(x) \) may be written

\[ U(x) = U_o + \frac{1}{2} x^2 \left\{ \frac{d^2U}{dx^2}|_o \right\} + \text{terms of order } x^3 + ... \]

The linear term \( x \left\{ \frac{dU}{dx}|_o \right\} \) does not appear - by the hypothesis that this is an equilibrium such that \( \frac{dU}{dx}|_o = 0 \). The terms in \( x^3 \) we are neglecting because \( x \) is small. We can further simplify by taking our arbitrary constant in the definition of \( U \) such that \( U_o = 0 \). Then we get the very simple \( U(x) = \frac{1}{2} x^2 \left\{ \frac{d^2U}{dx^2}|_o \right\} = \frac{1}{2} k x^2 \). (which defines a constant \( k \) This expression for \( U \) is like that for a spring.

**Solution for \( x(t) \)...**

We wish to solve the ODE

\[ m \frac{d^2x}{dt^2} = -kx(t) \quad \text{Valid for } x = \text{displacement from equilibrium SMALL} \]

corresponding to a potential energy \( U \) approximated near \( x = 0 \) as \( U \sim \frac{1}{2} k x^2 \) with \( k = U''(x_{equilibrium}) \)

The analysis a couple of pages back shows us that the velocity \( v = dx/dt \) is given in terms of the total energy and the position by

\[ v = \frac{dx}{dt} = \pm \sqrt{\frac{2 \left( E - U(x) \right)}{m}} = \pm \frac{1}{\sqrt{m}} \left( 2E - kx^2 \right) \quad \text{ ( } x = \text{displacement from equilibrium) } \]

or, (doing an *indefinite* integral)

\[ \int dt = \pm \int \frac{dx}{\sqrt{\left( 2E - kx^2 \right)/m}} + C \]

where \( E \) is knowable from initial conditions

\[ E = T_o + U_o = \left( 1/2 \right) m v_o^2 + \left( 1/2 \right) k x_o^2 \quad \text{(} x_o \text{being the initial displacement from equilibrium; it is small)} \]

We make the following definitions

\[ \omega = \sqrt{k/m}; \quad A = \sqrt{2E/k} \]
allowing us to write,

\[ v = \omega \sqrt{A^2 - x^2} \quad \text{or} \quad \frac{dx}{dt} = \omega \sqrt{A^2 - x^2} \quad \text{or} \quad \omega \, dt = \frac{dx}{\sqrt{A^2 - x^2}} \]

(the ± sign will sort itself out later) Then the integral becomes

\[ \omega t = \int \frac{dx}{\sqrt{A^2 - x^2}} + C \]

We can solve this by making the trig substitution \( x = A \sin \theta \) (or with more work, by looking it up in a table of integrals)

\[ \omega t - C = \int \frac{dx}{\sqrt{A^2 - x^2}} = \int \frac{A \cos \theta \, d\theta}{A \sqrt{1 - \sin^2 \theta}} = \int d\theta = \theta \]

ie. \( \theta = \omega t - C \). Now recalling the trig substitution that defined \( \theta \), we get

\[ x(t) = A \sin(\omega t - C) \]

A and C are essentially constants of integration that may be determined by referring to initial conditions.

\{Sanity checks: Is it reasonable that \( x(t) \) oscillates periodically? YES, that was predicted by our qualitative analysis. Is it dimensionally consistent? ie. does \( A \) have units of displacement? YES Does \( \omega \) have units of inverse time? YES\}

The period of the oscillations is \( \text{Period} = 2\pi / \omega = 2\pi / \sqrt{k/m} = 2\pi / \sqrt{U''(x)|_{x=eq} / m} \)

We note that the period does not depend on the initial conditions; it is purely a function of the system parameters, the mass and how the force varies with position.

An alternative way to solve the ODE \( m \frac{d^2x}{dt^2} = -k \, x(t) \) would be to follow the familiar rule for constant coefficient linear differential equations:Seek solutions which are exponential in time: \( x(t) = \exp(s \, t) \), and solve for \( s \). We’ll be taking that approach in the future.

At this point the lectures have covered material relevant to all of HW1A and also to HW1B.2 You are encouraged to attempt those problems now rather than wait til the middle of next week.

Case e) \( \ddot{F} = \bar{F}(\bar{v}) \) Again we restrict to 1-d (for now)

Just as in case (d), we cannot proceed the naïve way of slapping \( \int dt \) on both sides of \( m \, dv/dt = F(v) \). We’d get \( m \, v = \int F(v) \, dt \), which is true but not useful because we cannot perform the integral unless we already know \( v(t) \)
We do a separation of variables, where we put all the v's and dv's on one side and all the t's and dt's on the other:

\[ F(v) = m \ddot{v} = m \frac{dv}{dt} \quad \Rightarrow \]

\[ \int \frac{m}{F(v)} dv = \int dt \]

If we can analytically integrate the left side, and invert the resulting algebra, we get v as a function of time. One additional integration then gives x as a function of time.

**example**

The simplest example is linear viscous drag \( F = -cv \). The negative sign indicates that the force opposes the velocity. The definite integral becomes (in 1-d)

\[ -\int_{v_0}^{v} \frac{m}{cv} dv = \int_{0}^{t} dt \]

We define \( \kappa = c/m \), \{ \kappa has dimensions of \([F/V]/M = [ML/T^2] / [L/T] \) \} / \([M] = 1/[T] \). i.e., inverse time \} and find

\[ -\int_{v_0}^{v} \frac{dv}{v} = \ln(v/v_o) = -\kappa t \]

Thus

\[ v = v_o \exp(-\kappa t) \]

The speed diminishes exponentially with time. \{ sanity checks: does the formula have the right value at \( t = 0 \)? YES. Is it reasonable that the speed diminishes? YES. Is it reasonable that the speed asymptotes towards zero as \( t \rightarrow \infty \)? YES. is it dimensionally consistent? I.e. does \( \kappa \) have units of inverse time? YES Does increasing the mass influence the result in the right way? YES: Increased mass makes \( \kappa \) smaller, so the exponential rate of speed diminishing is slower which makes sense if there is more inertia. Similarly for diminishing the resistance \( c \). \}

A second integration will give us x(t):

\[ x(t) - x_o = \int_{t_0}^{t} v dt = v_o \int_{t_0}^{t} \exp(-\kappa t) dt = \frac{v_o}{\kappa} [1 - \exp(-\kappa t)] \]

**plot of x(t) for x_o = 0**

Interpretation: We observe that as \( t \rightarrow \infty \) this goes to an asymptotic value \( x_\infty = x_o + v_o/\kappa \)

The particle goes no more distance than \( v_o/\kappa \). \{ and takes infinite time to do so.\)
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1. A particle of mass \( m \) moves along a line \(-\infty < x < \infty\). It is subjected to a force \( f(t) = f_0 \cos(\alpha t) \)
It starts at time \( t = 0 \) at \( x = 0 \) with no velocity. Find a formula for its position \( x(t) \) at later times.

Solution: (using definite integrals)
\[
m \int dv = \int f(t) dt
\]
First integral:
\[
m [v - v_o] = \left( f_0 / \alpha \right) \sin(\alpha t) \bigg|_{t=0}^t = \left( f_0 / \alpha \right) \sin(\alpha t)
\]
so \( v(t) = \left( f_0 / m\alpha \right) \sin(\alpha t) + v_o \)

Second integral:
\[
\int v(t) dt = \int \left( f_0 / m\alpha \right) \sin(\alpha t) + v_o \] dt
\]
\[
x - x_o = \left[-\left( f_0 / m\alpha^2 \right) \cos(\alpha t) + v_o t \right]_0 = \left( f_0 / m\alpha^2 \right) \left[ 1 - \cos(\alpha t) \right] + v_o t
\]
However, \( x_o = 0 \) and \( v_o = 0 \) so
\[
x = \left( f_0 / m\alpha^2 \right) \left[ 1 - \cos(\alpha t) \right] + v_o t + x_o = \left( f_0 / m\alpha^2 \right) \left[ 1 - \cos(\alpha t) \right]
\]

Solution: (using indefinite integrals)
\[
m \int dv = \int f(t) dt = \int f_0 \cos(\alpha t) dt
\]
\[
m v = \left( f_0 / \alpha \right) \sin(\alpha t) + C_1
\]
\[
m \int dx = \int \left( f_0 / \alpha \right) \sin(\alpha t) + C_1 \] dt
\]
\[
x = -\left( f_0 / m\alpha^2 \right) \cos(\alpha t) + C_1 t + C_2
\]
Invoking initial condition that at time \( t=0 \), \( x = 0 \) tells us \( C_2 = \left( f_0 / \alpha^2 \right) \) (because \( \cos(0) = 1 \). Noteworthy: \( C_2 \) is related to the initial \( x \), but is not equal to it! Many students made the error of thinking \( C_2 = 0 \).
Invoking the condition that \( dx/dt = 0 \) at \( t = 0 \) tells us \( C_1 = 0 \) (because \( \sin(0) = 0 \)
\[
Thus \ x = \left( f_0 / m\alpha^2 \right) \left[ 1 - \cos(\alpha t) \right]
\]
Many students got \( x = -\left( f_0 / m\alpha^2 \right) \cos(\alpha t) \) … which does not satisfy the initial conditions.

2. A particle of mass \( m \) moves in 1-dimension and is subject to a force associated with the potential \( U(x) \) plotted here. The particle is released from rest at the illustrated position \( x_o \). Qualitatively describe its subsequent motion

Answer: At time 0 the particle is at \( x_o \) (indicated by vertical line) with zero speed, so its \( E \) is \( U(x_o) \) (indicated by horizontal line). At \( x_o \) it feels a force to the right. It moves that way, faster where \( U \) is low and and slower where \( U \) is high, but never stopping (because \( U(x) \) is always less than \( E \). Eventually the particle goes off towards \( x = +\infty \) at some constant speed.