All the systems pictured on the next page oscillate. But how can we quantitatively describe their motion? How are they similar or different? Let us explore some examples, starting with the mathematically simplest: the massive cart (without internal stiffness) and the exactly linear (and massless) spring, moving horizontally:

We choose $X(t)$ as our dynamical coordinate (it is often important to be clear about the precise definition of whatever time-dependent variable we use to describe the motion). In this case we use $X$, defined as the instantaneous length of the spring, i.e. the distance to the right that the cart is from the wall. Later we will find it convenient to change our coordinate.

**One way to get the differential equation** that governs $X(t)$ is to invoke $F = ma$. The mass's acceleration, in terms of $X$ is just $\frac{d^2X}{dt^2}$ towards the right. The force on the mass is, (according to our understanding of linear springs) $k \cdot$ its stretch, or $k \cdot (X(t)-L)$, towards the left. (We have had to introduce the natural length of the spring $L$. Interestingly, $L$ will drop out of most of our results.) We confirm that our formula tells us the force is to the left if $X > L$; this assures us that we have the sign of our formula correct.

We now write $F = ma$, or:

$$m \frac{d^2X}{dt^2} = -k \cdot (X(t) - L)$$

This is our ODE. The Simple Harmonic Oscillator (SHO)

**Another way to get the ODE** is to write an expression for the total energy

$$E = T + PE = \frac{1}{2} m \left(\frac{dX}{dt}\right)^2 + \frac{1}{2} k \cdot (X(t) - L)^2.$$  

where we have recalled the formula for the potential energy stored in a spring $(1/2) k \text{ stretch}^2$.

We also note that there are no energy dissipating or generating elements in this model, so $\frac{dE}{dt} = 0$.

Thus, using the chain rule:

$$0 = \frac{dE}{dt} = m \left(\frac{dX}{dt}\right) \left(\frac{d^2X}{dt^2}\right) + k \cdot (X(t) - L) \left(\frac{dX}{dt}\right)$$

$$= \left(\frac{dX}{dt}\right) \left[ m \left(\frac{d^2X}{dt^2}\right) + k \cdot (X(t) - L) \right]$$
The ODE is essentially the same for all of these oscillating systems. They are all of the form (simple harmonic Oscillator)

\[ M_{\text{eff}} \frac{d^2 z(t)}{dt^2} + K_{\text{eff}} z(t) = 0 \]

where \( z(t) \) is a measure of deviation away from equilibrium. \( z \) is our "dynamical coordinate."

Almost always for our applications the coefficients \( M_{\text{eff}} \) and \( K_{\text{eff}} \) will be positive and independent of \( t \) and \( z \). If yours aren't, you may have made a mistake.

For each physical system we must 1st choose the most useful coordinate \( z(t) \), and then do a bit of work to identify the values of \( M_{\text{eff}} \) and \( K_{\text{eff}} \).
If \( \frac{dE}{dt} \) is to be zero regardless of the speed of the mass, the part in square brackets (at the bottom of p 97) must be zero. …. which gives us the same ODE we have above.

We will sometimes find that our ODE is best derived by invoking \( F=ma \), or \( \text{Torque} = I \alpha \), or equivalent. On other occasions we will find it is easier to use an energy method, for example setting \( \frac{dE}{dt} = 0 \) if there is no dissipation.

======

**How to solve this ODE?**

It is worth noting that the ODE indicates that there is an equilibrium value of \( X \), such that \( F = 0 \) and \( \frac{dX}{dt} \) is zero. In the above example, that value is \( X_\text{eq} = L \). Let us shift coordinates to a quantity that represents deviation away from equilibrium: \( y(t) = X(t) - X_\text{eq} \).

Then \( \frac{d^2X}{dt^2} = \frac{d^2y}{dt^2} \) and \( X(t) - L = y \). The ODE for \( y(t) \) is then:

\[
m \frac{d^2y}{dt^2} = -k y(t)
\]

which is simpler than the ODE for \( X \).

This particular differential equation is very common, and can be expected to apply to a broad class of dynamical systems, especially if \( y \) is understood to represent small deviations away from equilibrium.

We know the solution,

\[
y(t) = A \cos \omega_\text{nt} + B \sin \omega_\text{nt}
\]

for any values of the constants \( A \) and \( B \), and for \( \omega_\text{natural} \) defined as \( (k/m)^{1/2} \).

A and \( B \) are not specified by the ODE. They can be determined by using additional information, for example initial conditions. It is, for example, easy to see that \( y \) (at \( t=0 \)) is just \( A \), and \( \frac{dy}{dt} \) at \( t = 0 \) is \( \omega_\text{n} B \).

A useful variation on the above form for \( y(t) \) is

\[
y(t) = C \cos(\omega_\text{n}t - \phi)
\]

\( C \) is the amplitude of the motion; \( \phi \) is its phase.

again with two apriori unspecified arbitrary constants \( C \) and \( \phi \). By direct substitution we can see that this also satisfies the ODE for any values of the constants \( C \) and \( \phi \).
There are simple relations between A and B and C and φ. One way to get them is to expand C cos{ω,t -φ} using the trig formula for the cosine of the sum of two angles:

\[ \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta. \]

\[ y(t) = C \cos(\omega t - \phi) = C \cos \phi \cos \omega t + C \sin \phi \sin \omega t \]

which permits us to identify

A = C \cos \phi and B = C \sin \phi. Also \( C = \sqrt{A^2 + B^2} \text{ and } \tan \phi = \frac{B}{A} \)

(but watch out for the \( + \pi \) ambiguity if you use \( \tan \phi = \frac{B}{A} \) to solve for \( \phi \).)

It is easy to see that

\( C = \text{the maximum value of } y, \text{ and } \)
\( C_{\omega n} = \text{maximum value of } \frac{dy}{dt}. \)

Note: frequencies \( \omega \) are in radians per second (or sometimes Avis) Frequencies \( f \) are cycles per second (or sometimes Hz) They correspond via \( f = \frac{\omega}{2\pi} \). Try not to confuse the two ways of quantifying frequency.

Here is a plot of a typical \( y(t) \). It corresponds to \( C = 1.7, \ \omega_n = 2 \text{ and } \phi = 1 \text{ radian. } \) It is also easy to see that it corresponds to \( A = \text{about } 0.92 \). B is a bit less obvious, but it can be worked out to be 1.43 by either using amplitude equal to about 1.7 = \( \sqrt{A^2 + B^2} \) \text{ or by using } \frac{dy}{dt} \text{ at } t = 0 \text{ (which is about 2.8 as could be seen by measurement off the graph) } = \omega_n B \)

The period can be measured off the plot to be 3.14 – the time between successive maxima, or twice the time between successive zeros. This is how we knew \( \omega_n \) ... Because period \( P = \frac{2\pi}{\omega_n}. \)

\[ y = 1.7 \cos(2t-1) = \]
\[ 0.92 \cos 2t + 1.43 \sin 2t \]

Let us write expressions for the PE using \( y = C \cos(\omega t - \phi) \)

100
\[ PE = \frac{1}{2} k y^2 = \frac{1}{2} k C^2 \cos^2(\omega_n t - \phi) \]

which oscillates but is always positive. Its maximum value is \( k C^2/2 \) corresponding to the PE the system has when \( y \) is at its maximum value \( C \) and \( \frac{dy}{dt} \) is zero.

Similarly we may construct the Kinetic energy

\[ T = \frac{1}{2} m (\frac{dy}{dt})^2 = \frac{1}{2} m \omega_n^2 C^2 \sin^2(\omega_n t - \phi) \]

which oscillates but is always positive. Its maximum value is \( \frac{1}{2} m \omega_n^2 C^2 = k C^2/2 \). This is equal to the maximum PE but maximum KE occurs at a different time, when \( y = 0 \) and \( \frac{dy}{dt} \) is maximum.

The total energy is \( E = T + PE = \)

\[ E = \frac{1}{2} m \omega_n^2 C^2 \sin^2(\omega_n t - \phi) + \frac{1}{2} k C^2 \cos^2(\omega_n t - \phi) = k C^2/2 = PE_{\text{max}} = KE_{\text{max}} \]

This is a constant, as it ought be.

→ We note, the energy of the motion depends on the amplitude of the motion \( C \)

→ The period \( P \) of the motion \( 2\pi/\omega_n \) is independent of the amplitude and energy. This makes such oscillators good clocks, the rate at which they mark out time won't depend on how energetic their motion is.

→ The oscillation frequency \( \omega_{\text{natural}} \) scales with the (square root of) the stiffness:

greater stiffness: shorter period

→ The oscillation frequency \( \omega_{\text{natural}} \) scales with the inverse of the (square root of) the mass:

more inertia: longer period

==========

The above analysis applies to the case of the rigid cart and the massless ideal spring moving horizontally. We get the same motion, qualitatively, for other systems….. For example the cantilevered beam for which we can corroborate the predicted mass dependence of the natural frequency with a simple demonstration.

\[ P \propto m^{1/2} \]

--------------------

101
Consider another system for which we derive the same ODE: Let us put the cart in a parabolic bowl whose bottom is a distance S from the wall.

We will continue to write the spring's PE as \((1/2) k (X-L)^2\) (assuming the spring is very long and we needn't fret about its angle) If the bowl has parabolic profile, height \(= \alpha (S-X)^2\), then the PE due to gravity is \(\alpha mg (S-X)^2\). Total PE is then 
\[ \frac{1}{2} k (X-L)^2 + \alpha mg (S-X)^2. \]

Equilibrium is at \(d\ PE/ dX = 0\),

i.e at \(X_{eq} = \frac{(kL + 2\alpha mg S)}{(k + 2\alpha mg)}\)

The PE is a quadratic function of \(X\), so it may be re-written

\[ PE = PE(X_{eq}) + \frac{1}{2} \left( \frac{d^2 PE}{dX^2} \right) (X - X_{eq})^2 \]

( and we note that \(d^2 PE / dX^2\),
which we define as \(k_{\text{effective}}\), is \(k + 2\alpha mg\) )

The Kinetic energy due to horizontal motion is 
\((1/2) m (dX/dt)^2\). It is not necessary to make the simplification that the Kinetic energy due to vertical motion is negligible, but let us keep it simple for now by assuming \(\alpha\) is sufficiently small that we can neglect it.

Now \(E = T + PE = (1/2) m (dX/dt)^2 + (1/2) k_{\text{effective}} (X(t) - X_{eq})^2 + PE(X_{eq})\)

( the last term is a constant and usually uninteresting )
( the precise value of \(X_{eq}\) is often uninteresting also)

One proceeds as before and finds an ODE for \(X\):

\[ m \left( \frac{d^2 X}{dt^2} \right) + k_{\text{eff}} (X(t) - X_{eq}) = 0 \]

The equation governing \(y(t)\) defined as \(X(t) - X_{eq}\) is

\[ m \left( \frac{d^2 y}{dt^2} \right) + k_{\text{eff}} y(t) = 0 \]

The equilibrium position has been shifted by the introduction of the bowl. The cart now sits at some point on the side of the bowl. The effective stiffness has been changed too,
but the qualitative behavior is unchanged:

\[ \Rightarrow \text{Deviations away from equilibrium vary sinusoidally in time,} \]
\[ \text{with a period that is independent of amplitude} \]

In the HW you will find, when you hang the mass and spring from the ceiling instead of having them be horizontal, that the equilibrium is shifted, but the \( k_{\text{eff}} \) is unchanged.

If we had two springs:

with natural lengths \( L_1 \) and \( L_2 \)

The PE becomes

\[ \text{PE} = \frac{1}{2} k_1 (X - L_1)^2 + \frac{1}{2} k_2 (S - X - a - L_2)^2 \]

This is quadratic in \( X \). It is therefore writable as

\[ \text{PE} = \text{PE}(X_{\text{eq}}) + \frac{1}{2} k_{\text{eff}} (X - X_{\text{eq}})^2 \]

where \( k_{\text{eff}} \) is found from the second derivative of PE, ie.

\[ k_{\text{eff}} = k_1 + k_2. \]

And where the precise value of \( X_{\text{eq}} \) could be found using a bit of algebra. Without doing any further analysis, we see that adding the second spring has shifted the equilibrium, (finding the equilibrium point is often tedious and usually uninteresting) and changed the effective stiffness; the stiffnesses of the individual springs simply add.

The ODE for deviations away from \( X_{\text{eq}} \) is

\[ m \frac{d^2y}{dt^2} + (k_1 + k_2) y = 0 \]

The natural frequency is \( \omega_{\text{nat}} = \left( \frac{k_1 + k_2}{m} \right)^{1/2} \)

The period of the motion is \( 2 \pi \left[ \frac{(k_1 + k_2)}{m} \right]^{1/2} \)

Simple pendulum
A massless rod, a point bob of mass m, in a uniform g field.

We choose the dynamical coordinate \( \theta(t) \) in radians away from hanging down to describe the motion of the pendulum.

We could derive the ODE from torque = \( I\alpha \), but it is more elegant, and more consistent with the latter part of the course, to construct \( E \) and then set its \( \frac{d}{dt} \) to 0.

Kinetic energy is (because the speed of the mass is \( L \frac{d\theta}{dt} \))

\[
T = \frac{1}{2} m \left( L \frac{d\theta}{dt} \right)^2
\]

Potential energy is \( mgh \), where \( h \) is \( -L \cos \theta \) is the height below the ceiling (we have chosen the ceiling as our zero of potential energy). We could have chosen any reference point, like the bottom at \( \theta = 0 \) for example, in which case we'd get \( PE = mgL(1-\cos \theta) \) which differs merely by a constant. It doesn't much matter, and we usually merely make a choice that is convenient.

\[
PE = -mgL \cos \theta
\]

Then \( \frac{dE}{dt} = \frac{d(T+PE)}{dt} = 0 = \frac{d\theta}{dt} \{ m L \frac{d\theta}{dt} + mgL \sin \theta \} \)

The ODE is found by setting the part in \( \{ \} \) to zero.

\[
m L^2 \frac{d^2\theta}{dt^2} + mgL \sin \theta = 0
\]

This is almost but NOT exactly the advertised ubiquitous form. \( M_{\text{eff}} \frac{d^2x}{dt^2} + K_{\text{eff}} x = 0 \)

We can recover the expected form by invoking the assumption \( \theta = \text{small} \) (and that \( \theta \) is measured in radians – which we assumed anyway in our form for Kinetic energy). The condition \( \theta = \text{small} \) may be checked after we know what values of \( \theta \) are realized. Then

\[
m L^2 \frac{d^2\theta}{dt^2} + mgL \theta = 0
\]

Now it is easy to identify \( M_{\text{eff}} \) with \( mL^2 \) and \( K_{\text{eff}} \) with \( mgL \), and \( \theta \) as our dynamical coordinate.

Then we deduce \( \omega_n = \sqrt{\frac{g}{L}} \) and

and

\[
\theta (t) = A \cos \omega_n t + B \sin \omega_n t
\]

with \( A = \theta(t=0) \) and \( B \omega_n = \frac{d\theta}{dt} \) at \( t=0 \).

Or we could write \( \theta (t) = C \cos (\omega_n t - \phi) \)
The period $2 \pi \left[ \frac{g}{L} \right]^{1/2}$ is independent of the mass.

*Do not confuse* $d\theta/dt$ the angular speed of the pendulum in space, which is a geometric quantity, with $\omega_n$ which is an angular speed of the phase of the oscillator in a more abstract space.

Our assumption $\theta \sim$ small will be satisfied if the initial conditions are not too severe. If $\theta$ and $d\theta/dt$ start out small, this solution indicates that they remain small.

How small is small enough? It depends on the accuracy you need. At $\theta = 30$ degrees which is about 0.5 radians, $\sin \theta$ and $\theta$ differ by only 4%. At 5 degrees they differ by 0.12%

**Why is this ODE so common?**

I've made the assertion that we should expect for a governing ODE

$$M_{\text{eff}} \frac{d^2 z}{dt^2} + K_{\text{eff}} z = 0$$

with $M_{\text{eff}}$ and $K_{\text{eff}}$ positive constants independent of $z$ and $t$, and with $z$ representing small deviations away from static equilibrium.

Why is it to be expected?

A partial answer is because, in any 1-degree of freedom system with an equilibrium at all, we should expect a PE proportional to $(X(t) - X_{eq})^2$ plus higher order terms. Because…

Generically PE must be parabolic near its minimum. Doing a Taylor series in $X - X_{eq}$:

$$PE = PE_{\text{equilibrium}} + \left(\frac{1}{2}\right) PE''_{\text{equilibrium}} (X(t) - X_{eq})^2 + .....$$

For $y = X - X_{eq}$ small, we write

$$E_{\text{total}} = \left(\frac{1}{2}\right) M (dy/dt)^2 + \left(\frac{1}{2}\right) K_{\text{eff}} y^2 + PE_{\text{equilibrium}}$$

where $K_{\text{eff}}$ is *defined* as the second derivative of PE evaluated at $X_{eq}$.

BY invoking $dE/dt = 0$, our ubiquitous ODE is then obtained, and seen to apply for sufficiently small deviations $y$. 


A cart of mass $m$ is constrained by three springs of different stiffnesses, as pictured.

What is the period of oscillation of the cart? Hint: what is the effective stiffness?

You might find it useful to draw a free body diagram for the mass for a state in which it is displaced a small amount to the right of its equilibrium: How much does the force in each spring change and what is the net change of force on the mass.

You may also find it useful to consider your expression for various limiting cases, such as one of the springs having zero stiffness.

Answer: The three stiffnesses just ADD. You can see this in detail by recognizing that the PE is 
\[
\frac{1}{2} k_1 (X-L_1)^2 + \frac{1}{2} k_2 (S-a-X-L_2)^2 + \frac{1}{2} k_3 (S-a-X-L_3)^2,
\]
whose $X^2$ coefficient is the effective stiffness is \( \frac{1}{2} (k_1+k_2+k_3) \).

Sanity check: make sure the effective stiffness is always positive.

You can also see this by imagining a mass displacement $\delta X$ to the right of equilibrium. The first spring is then stretched and provides a force (over and above what it provides at equilibrium) to the left by an amount $k_1 \delta X$. The other springs are compressed by $\delta X$ and provide forces (over and above what the provide at equilibrium) to the left of $k_2$ and $k_3$ times $\delta X$.

So the natural frequency $\omega_n$ is \( \sqrt{\frac{(k_1+k_2+k_3)/m}{1/2}} \) and the period is $2\pi/\omega_n$.

NB: The way the springs combine is not the way Ohmic resistors in parallel combine, even though the sketch looks much the same for springs and resistors.

NB: Springs in series would combine like effective stiffness = \( \frac{1}{(1/k_1 + 1/k_2)} \).

For this system the natural frequency would be $\omega_n = \sqrt{m \left( \frac{1}{1/k_1 + 1/k_2} \right)}^{-1/2}$ and the period is $2\pi/\omega_n$. 