1. **30 points** A satellite is in a circular orbit (labeled 1) around the earth (mass M) at radius R with speed $v_{\text{initial}} = \sqrt{GM/R}$. We wish it to climb to a higher circular orbit, at radius $R' > R$ with speed $v_{\text{final}} = \sqrt{GM/R'}$. We will do this by means of an elliptical Hohmann transfer orbit. It applies an impulse (with a change of speed $\Delta v$) at time zero, which puts it into the transfer orbit (labeled 2). On arriving at the new distance $R'$ it is at the apogee of the transfer orbit. It then applies an additional impulse $\Delta v'$, increasing its speed again, to enter the final circular orbit (labeled 3).

What is the angular momentum $\ell$ and energy $e$ (per mass) of the transfer orbit? [This is readily found in terms of R and R', and knowing the transfer orbit's formula for r(\phi); see below.]

What are the needed instantaneous changes of velocity $\Delta v$ and $\Delta v'$?

All answers should be in terms of M, G, R and R'.

To answer these questions you may find it useful to recall the formula

$$r(\phi) = (\ell^2/GM) \left[ 1 + \left( 1 + \frac{2e}{\ell^2/GM^2} \right)^{1/2} \cos(\phi) \right]^{-1} = a / \left[ 1 + e \cos \phi \right]$$

Hint: what are $a$ and $e$ in terms of R and R'? Note that e must be negative.

2. **15 points** Two black holes, of masses 2 $M_{\text{solar}}$ and 1 $M_{\text{solar}}$ respectively, are in circular orbits around each other. Their separation is $d = 10,000$ km. They each have a speed relative to the joint center of mass (indicated in the figure by a *).

- What is the ratio of their speeds relative to the center of mass? Which is moving faster?
  (hint: think about total system momentum)

- What is the period of their orbit in seconds?

- Where is each black hole at a time 1/4 period later than the time in the figure? Indicate these positions in the figure.

  P.S. $M_{\odot} = 2 \times 10^{30}$ kg  $G = 6.67 \times 10^{-11}$ N m$^2$/kg$^2$.

3. **15 points** A spaceship is in a circular orbit above a black hole of one solar mass. The orbit has a radius D.  
- In terms of G, $M_{\text{solar}}$ and D, what is the spaceship's acceleration at this moment? 
- What is the period P of its orbit?  
- Evaluate P numerically (you may use a calculator) for D = 5000 km, $M_{\odot} = 2 \times 10^{30}$ kg; $G = 6.67 \times 10^{-11}$ N m$^2$/kg$^2$.

- What is the tidal effect on a particle floating in the spaceship? I.e., given a particle a distance h to the right of the spaceship's center of mass (with h << D), what is the particle's acceleration relative to the ship's center of mass? Evaluate this numerically (you may use a calculator) for D = 5000 km, $h = 1$ meter, $M_{\odot} = 2 \times 10^{30}$ kg; $G = 6.67 \times 10^{-11}$ N m$^2$/kg$^2$. 
