Physics 325 Discussion 1 Solutions

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January 15, 2019

1 Problem 1

The helical path of the bee is given by the equation,

\[ \mathbf{r}(t) = b \sin(\omega t) \hat{i} + b \cos(\omega t) \hat{j} + ct \hat{k} \]  

(1)

The acceleration is determined as follows,

\[ \mathbf{a}(t) = \frac{d^2 \mathbf{r}}{dt^2} = -\omega^2 b \sin(\omega t) \hat{i} - \omega^2 b \cos(\omega t) \hat{j} + 0 \hat{k} \]  

(2)

The magnitude is given by,

\[ ||\mathbf{a}(t)|| = \sqrt{a_x(t)^2 + a_y(t)^2 + a_z(t)^2} \]  

which is simply just,

\[ ||\mathbf{a}(t)|| = \sqrt{(-\omega^2 b \sin(\omega t))^2 + (-\omega^2 b \cos(\omega t))^2 + (0)^2} = \omega^2 b \sqrt{\sin^2(\omega t) + \cos^2(\omega t)} = \omega^2 b \]  

(4)

2 Problem 2

(a) When \( x \ll 1 \), the following are the Taylor series expansions of \( \cos(x) \) and \( \sin(x) \),

\[ \cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \]  

\[ \sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \]  

(5)

The two non-vanishing leading terms in the following expressions are as follows,

\[ \frac{1 - \cos(x)}{x^2} \approx \frac{1}{2} - \frac{x^2}{24} \]  

\[ \frac{\sin(x)}{x} \approx 1 - \frac{x^2}{6} \]  

(6)

The following integrals are to be evaluated,

\[ I_1(a) = \int_{0}^{a} \frac{1 - \cos(x)}{x^2} \, dx \]  

\[ I_2(a) = \int_{0}^{a} \frac{\sin(x)}{x} \, dx \]  

(7)
for $a \ll \pi$. Using the Taylor series expansions above, these integrals can be evaluated to second order,

$$I_1(a) \approx \int_0^a \left(\frac{1}{2} - \frac{x^2}{24}\right) \, dx = \left[\frac{1}{2} x - \frac{x^3}{72}\right]_0^a = \frac{a}{2} - \frac{a^3}{72}$$

$$I_2(a) \approx \int_0^a \left(1 - \frac{x^2}{6}\right) \, dx = \left[x - \frac{x^3}{18}\right]_0^a = a - \frac{a^3}{18}$$

(8)

(b) For $a = 1$, (8) becomes,

$$I_1(1) \approx \frac{1}{2} - \frac{1}{72} = \frac{35}{72}$$

$$I_2(1) \approx 1 - \frac{1}{18} = \frac{17}{18}$$

(9)

3 Problem 3

The vector field is,

$$\mathbf{F}(x, y, z) = x^3z^4\hat{i} + xyz^2\hat{j} + x^2y^2\hat{k}$$

(10)

(a) The divergence of $\mathbf{F}$, $\nabla \cdot \mathbf{F}$ is given by,

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \frac{\partial}{\partial x}(x^3z^4) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(x^2y^2) = 3x^2z^4 + xz^2 + 0 = 3x^2z^4 + xz^2$$

(11)

(b) The curl of $\mathbf{F}$, $\nabla \times \mathbf{F}$ is given by,

$$\nabla \times \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) \hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) \hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \hat{k}$$

(12)

which gives,

$$\nabla \times \mathbf{F} = \left(\frac{\partial}{\partial y}(x^2y^2) - \frac{\partial}{\partial z}(xyz^2)\right) \hat{i} + \left(\frac{\partial}{\partial z}(x^3z^4) - \frac{\partial}{\partial x}(x^2y^2)\right) \hat{j} + \left(\frac{\partial}{\partial x}(xyz^2) - \frac{\partial}{\partial y}(x^3z^4)\right) \hat{k}$$

(13)

This simplifies to,

$$\nabla \times \mathbf{F} = (2x^2y - 2xyz)\hat{i} + (4x^3z^3 - 2xy^2)\hat{j} + yz^2\hat{k}$$

(14)

(c) Therefore,

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial}{\partial x}(2x^2y - 2xyz) + \frac{\partial}{\partial y}(4x^3z^3 - 2xy^2) + \frac{\partial}{\partial z}(yz^2) = 4xy - 2yz + 0 - 4xy + 2yz = 0$$

(15)
4 Problem 4

The height of the hill \( z(x, y) \) is given by,

\[
z(x, y) = \eta(2xy - 3x^2 - 4y^2 - 18ax + 28ay + 12a^2)
\]  

(16)

where \( \eta = 0.01 \text{ m}^{-1} \) and \( a = 1 \text{ m} \). You can see that \( z(x, y) \) has dimensions of meters as it should because each term within in the parentheses has dimension \( \text{m}^2 \) while \( \eta \) has dimensions of \( \text{m}^{-1} \) so when multiplied altogether, this gives dimensions of \( \text{m} \) or meters.

(a) The top of the hill \( (x_0, y_0) \) is located at a maximum. This can first be determined by where the partial derivatives vanish,

\[
\frac{\partial z}{\partial x} \bigg|_{(x_0, y_0)} = 0
\]

\[
\frac{\partial z}{\partial y} \bigg|_{(x_0, y_0)} = 0
\]

Therefore,

\[
\eta(2y_0 - 6x_0 - 18a) = 0 \rightarrow 3x_0 - y_0 = -9a
\]

\[
\eta(2x_0 - 8y_0 + 28a) = 0 \rightarrow x_0 - 4y_0 = -14a
\]

Solving this system of equations gives \( (x_0, y_0) = (-2a, 3a) = (-2, 3) \). Hence, \( (-2a, 3a) = (-2, 3) \) is the top of the hill.  

(b) The gradient of (16) is,

\[
\nabla z(x, y) = \frac{\partial z}{\partial x} \hat{i} + \frac{\partial z}{\partial y} \hat{j} = \eta((2y - 6x - 18a)\hat{i} + (2x - 8y + 28a)\hat{j})
\]

(19)

At \( (x, y) = (1, 1) \) the gradient is,

\[
\nabla z \bigg|_{(1,1)} = 22\eta(-\hat{i} + \hat{j})
\]

(20)

The magnitude of the gradient \( |\nabla z|_{(1,1)} \) is given by,

\[
|\nabla z|_{(1,1)} = \sqrt{\left( \frac{\partial z}{\partial x} \bigg|_{(1,1)} \right)^2 + \left( \frac{\partial z}{\partial y} \bigg|_{(1,1)} \right)^2} = 22\sqrt{2}\eta = 0.22\sqrt{2}
\]

(21)

The angle the hill surface makes with horizontal is then simply,

\[
\theta = \arctan \left( \frac{|\nabla z|_{(1,1)}}{\partial z/\partial x_{(1,1)}} \right) = \arctan(0.22\sqrt{2}) \approx 17.28^\circ
\]

(22)

(c) Since \( \nabla z \bigg|_{(1,1)} = 22\eta(-\hat{i} + \hat{j}) \), the ball will want to roll downhill which is in the direction of \( -\nabla z \bigg|_{(1,1)} = 22\eta(\hat{i} - \hat{j}) \), which is southeast.

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1It’s good practice to check if this point is indeed a maximum. You can use the second-derivative (or “saddle” point) test to ensure this. This involves computing \( H = \left( \frac{\partial^2 z}{\partial x^2} \bigg|_{(x_0,y_0)} \right) \left( \frac{\partial^2 z}{\partial y^2} \bigg|_{(x_0,y_0)} \right) - \left( \frac{\partial^2 z}{\partial x \partial y} \bigg|_{(x_0,y_0)} \right)^2 \). Since \( \frac{\partial^2 z}{\partial x^2} \bigg|_{(-2a,3a)} = -6\eta, \frac{\partial^2 z}{\partial y^2} \bigg|_{(-2a,3a)} = -8\eta, \) and \( \frac{\partial^2 z}{\partial x \partial y} \bigg|_{(-2a,3a)} = 2\eta \), it follows that \( H = 44\eta^2 > 0 \). This ensures that \( (-2a, 3a) = (-2, 3) \) is either a maximum or a minimum. However, since \( \frac{\partial^2 z}{\partial x^2} \bigg|_{(-2a,3a)} = -6\eta < 0 \) (or if you’d like, \( \frac{\partial^2 z}{\partial y^2} \bigg|_{(-2a,3a)} = -8\eta < 0 \)), this is in fact a maximum.
We are given $b \cdot v = \lambda$ and $b \times v = c$. Using the hint, notice the following,

\[
b \times c = b \times (b \times v) = (b \cdot v)b - (b \cdot b)v = \lambda b - (b \cdot b)v
\]  

(23)

This means,

\[
(b \cdot b)v = \lambda b - (b \times c)
\]  

(24)

From this, one can solve for $v$ easily,

\[
v = \frac{\lambda b - (b \times c)}{(b \cdot b)}
\]  

(25)