The Procedure for Multidimensional Integration $\int_{\mathbb{R}^n} F(r_i) \otimes d\mathbb{R}^n$

Definition of the essential word "**PARAMETRIZE**" as used in this context:

> Express **all quantities that vary over the integral** in terms of your **integration parameters** (**IP**s) & constants.

1. Parametrize the **Region $\mathbb{R}^n$**

- **a.** Pick your **coordinate system** $\boldsymbol{r_i} = (x,y,z)$ Cartesian, or (r,θ,ϕ) spherical, or (s,ϕ,z) cylind.
- **b.** Pick your n **integration parameters** $\boldsymbol{u_j}$ – a.k.a. IPs or "**sweeping parameters**" – that will sweep out the region $\mathbb{R}^n$. If possible, use one or more of your chosen coordinates r_i.
- **c.** Describe the *shape* of $\mathbb{R}$ by expressing your coordinates r_i as **functions**[1] $\boldsymbol{r_i(u_j)}$ of the IPs
- **d.** Describe the *edges* of $\mathbb{R}$ by providing **bounds** on each integration parameter u_j

2. Parametrize the **Differential $d\mathbb{R}^n$** using your coordinate system's **Line Element** $d\vec{l}$

Method 1: Visualization

$\boldsymbol{d\mathbb{R}^n} = d\vec{l}\,|d\vec{A}|dV$ is **how much space** (length | area | volume) **you sweep out when you increase every IP $\boldsymbol{u_j}$ by $\boldsymbol{du_j}$.** Figure it out with a sketch and/or the line element $d\vec{l}$ of your coordinate system. This method works best when the integration parameters u_j are actual coordinates r_j.

Method 2: Formalism

Defining $d\vec{l}_u \equiv \dfrac{\partial \vec{l}}{\partial u}\, du$,

$$d\vec{l}_{\text{path}} = d\vec{l}_u$$
$$d\vec{A} = d\vec{l}_u \times d\vec{l}_v$$
$$dV = \left(d\vec{l}_u \times d\vec{l}_v\right) \cdot d\vec{l}_w$$

3. Construct the **Integral** expressing *everything* in terms of your **IPs** and **constants**

Use your **coordinate functions $\boldsymbol{r_i(u_j)}$** from to express *everything that varies* in the integrand *entirely* in terms of the IPs and constants. Watch out especially for spher/cylind unit vectors!

Your integral must be **doable** = something you can type into Wolfram Integrator[2], and must **make sense** = give a result that depends only on quantities that *survive the integration*. (Example of nonsense: a final result with an IP left in it!) Proper integrals have this form:

$\mathbb{R}^1$ path integral	$\mathbb{R}^2$ surface integral	$\mathbb{R}^3$ volume integral
$\int_{u_i}^{u_f} G(u)\, du$	$\int_{v_i}^{v_f}\int_{u_i}^{u_f} G(u,v)\, du\, dv$	$\int_{w_i}^{w_f}\int_{v_i}^{v_f}\int_{u_i}^{u_f} G(u,v,w)\, du\, dv\, dw$

For **vector integrals**, you will get one such scalar integral **per component**.

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[1] What to call these functions $r_i(u_j)$? **Constraint functions** is a good name, as that's what they do: constrain the coordinates to lie on your region $\mathbb{R}$. I like the descriptive **shape functions**, but we'll go with **coordinate functions**.

[2] Free integration available online at http://integrals.wolfram.com (indefinite integrals only). The new, insanely powerful WolframAlpha can do definite integrals too → see http://wolframalpha.com/examples/Calculus.html

Tips and Tricks for Multi-D Integration

➔ Choose the coordinate system r_i that best matches the integration region $\mathbb{R}$, not the integrand.

➔ If your integral gives a vector result, you must split it into 3 separate integrals, one for each component. (Why? Vectors sum by components, and integrals are just that: sums.)

➔ Beware of non-Cartesian unit vectors in your integrand! If $\hat{r}$, $\hat{s}$, $\hat{\theta}$, or $\hat{\phi}$ appear in your integrand and are associated with coordinates over which you're integrating, you *cannot* pull them out of the integral because they're *not constant* → transform them to fixed, Cartesian unit vectors before you integrate. The one exception is in field integrals (next point): if $\hat{r}$, $\hat{s}$, $\hat{\theta}$, or $\hat{\phi}$ are associated with the *field-point* coordinates rather than the source-point coordinates, they *are* constant over the integral and can be left alone.

➔ To change the direction of a path integral, change the bounds, not $d\vec{l}$. Do not mess with the direction of $d\vec{l}_{\text{path}}$: it is tied to your coordinate system and your parametrization of the path by the strict formula $d\vec{l}_{\text{path}} = (d\vec{l}\,/\,du)\,du$. In contrast, you are completely free to choose the order of the bounds on your IP, u. Final point: if you do *both*, you'll have done *nothing* → try it and see!

➔ If your integral gives you a field as a result (i.e., a function like $V(\vec{r})$ or $\vec{E}(\vec{r})$) you must be careful to distinguish between **field point** and **source point** coordinates:

> *The source-point coordinates* $\vec{\boldsymbol{r}}_q$ *vary over the integral, while*
> *the field-point coordinates* $\vec{\boldsymbol{r}}$ *do not and can be treated as constants.*

Be sure to label them differently: use a subscript or prime to identify the source point coordinates in your expressions, and use different symbols on your sketches such as

- • for the source point (because it looks like a physical charge … to me anyway ☺)
- × for the field point (because it reminds me of a treasure map. seriously.)

➔ Always consider the **symmetries** of the system, namely transformations that leave the system unchanged. If a system has such symmetries, then any field it produces or quantity that describes it will *also* be unchanged under those transformations. This allows you to simplify your work in advance! For integrals producing fields, symmetries can restrict the functional dependence of the result. For integrals producing vectors (constant vectors or vector fields), symmetries can restrict the number of components you have to calculate.

➔ Be sure to shift and/or rotate your coordinate system in order to match its symmetries to those of your region of integration. Note that the *z-axis* is the axis of symmetry for both the cylindrical and spherical coordinate systems, while the *origin* is the point of symmetry for all coordinate systems. Detail to keep in mind: if you shift/rotate your coordinate system to get a nice description of $\mathbb{R}$, you must transform your *integrand too*, in the same way.