Physics 325 – Homework #9

Summary of Variational Calculus: If you want to extremize a quantity $S$ that is integrated over some path \{q_i(t)\} of your system, and $S$ is described by the integral

$$S = \int L(q_i(t), \dot{q}_i(t), t) \, dt$$

with fixed endpoints $t_1, q_i(t_1)$, and $t_2, q_i(t_2)$,

then the path \{q_i(t)\} that extremizes this integral satisfies the Euler-Lagrange equation

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0$$

for each coordinate $q_i$.

Finally, the Hamiltonian $H \equiv \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$ is constant if $t$ is cyclic because $\frac{dH}{dt} = -\frac{\partial L}{\partial t}$.

That’s it! Working with these equations boils down to identifying which parameters of your system are:

- the independent variable $t$ that serves as your parameter of integration,
- the (remaining) coordinates $q_i(t)$ describing the state of the system (I say “remaining coordinates” since the variable of integration, $t$, is often chosen to be one of the coordinates)
- the integrand $L(q_i, \dot{q}_i, t)$, called the Lagrangian of the problem

Solving calculus-of-variations problems is thus a 3-step game:

1. Set up the integral $S$ you are trying to extremize (if isn’t just given to you).
2. Play “match the letters” $\rightarrow$ what’s “$t$”? what are the “$q_i$”? what’s “$L$”? 
3. Write down the E-L equation for each $q_i$, then solve them for the path \{q_i(t)\} that extremizes $S$.

You may use [http://wolframalpha.com](http://wolframalpha.com) (or any equivalent tool) to evaluate any of the integrals on this homework. You must set up the integrals yourself, but some are fairly complicated, so let’s get some more practice with integration software. Please review the Homework 2 Appendix : Mathematica Oddities, particularly for the later problems.

Problem 1 : Simple Euler-Lagrange

(a) Find the path $y(x)$ for which the integral $S = \int \sqrt{x} \sqrt{1 + (y')^2} \, dx$ is stationary (i.e. is extremized) between fixed endpoints, and sketch its shape.

**DOT-NOTATION NOTE:** Since the coordinates of your path ($q_i$) depend on only one parameter ($t$) it is common to use dot notation to denote the derivative $\dot{q}_i \equiv dq_i / dt$ even if your independent variable is not time. If you feel uncomfortable with this, feel free to rewrite the integral in this problem as $\int \sqrt{x} \sqrt{1 + y'^2} \, dx$. The point is: there is no possible confusion between $y'$ and $\dot{y}$ since $y$ is only a function of one variable $x$ = the one independent variable you’re using as your integration parameter. (A path always has only one parameter of integration, and the Euler-Lagrange equations only apply to path integrals.)

(b) Using your skill at “playing with differentials”, rewrite the integral $S$ to make $y$ the independent variable.

**STRATEGY:** An important strategic issue that arises all the time in variational problems is what independent variable should you choose? The choice you make is surprisingly important, and here’s why:

E-L equation: $\frac{\partial L}{\partial q} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \leftarrow$ full derivative on right-hand side can be horrible!

The right-hand side is often a horrible mess to compute, and it produces an ODE that is second-order in $t$ (i.e., contains $\dot{q}$). The equation is greatly simplified if you can arrange things so that $q$ does not appear explicitly in $L$. That makes the left-hand side zero, and the equation becomes $\partial L / \partial \dot{q} = C$ (a constant). This is
a first-order ODE, which is usually much simpler to solve. As we discussed in lecture, any coordinate \( q \), that does not explicitly appear in the Lagrangian is called a cyclic or ignorable coordinate. Of course there are always exceptions. If you want \( y(x) \), but decide to solve for \( x(y) \) to simplify the E-L equation, the final step of inverting \( x(y) \) may occasionally be so tricky that the original procedure may actually be easier.

(c) In part (a), the coordinate \( y \) was cyclic and this greatly simplified the E-L equation to 1\(^{st}\)-order form. Your part (b) Lagrangian does not have this property: the (dependent) coordinate is now \( x \), and it does appear in \( L \rightarrow \) not cyclic. If you apply the E-L equation to your new Lagrangian, you will get a horrible mess that will produce a 2\(^{nd}\) order ODE that is nearly impossible to solve. (Try it for a couple of minutes to see what I mean by “horrible mess”.) However, the independent coordinate \( y \) is now cyclic, so you have an alternative: the Hamiltonian \( H \) is now constant. Construct \( H \), solve the 1\(^{st}\)-order ODE “\( H \) = constant”, and verify that you get the same path as in part (a). You must, as you are extremizing the exact same integral \( S \)!

Problem 2 : Fermat’s Oasis

The air above a hot road is less dense near the road and more dense as you get higher above the road. This results in a refractive index that varies with the height \( y \) above the road : \( n(y) = n_0 \sqrt{1 + y/a} \) where \( n_0 \) and \( a \) are constants and \( y=0 \) is the road. Fermat’s principle states that the path light travels between two fixed points is the path that minimizes the total travel time. Obtain the Euler-Lagrange equation for the paths that light can take in this heated air. Instead of finding its general solution, show that the particular path \( y(x) = x^2 / 4a \) solves it.

FYI: This path is a parabola that touches the road. It shows how light from the blue sky can "bounce" off the hot road and hit your eyeball while you are walking along. Your eye will interpret this as something blue on the road off in the distance, which is the familiar "oasis mirage" of water apparently lying ahead on a hot surface.

STRATEGY: This problem is a perfect opportunity to practice the technique of choosing your independent variable wisely. I urge you to try both alternatives: use \( x \) as your independent variable, see what Lagrangian you get, and set up the E-L equation … then repeat the procedure using \( y \) as your independent variable. In one case you will get a Lagrangian of the form \( L(q, \dot{q}) \) and in the other case you will get the form \( L(q, t) \).

See which E-L equation is easier to solve! This strategy issue appears again in the later problems, especially in problem 4 where good strategy is essential.

Problem 3 : Maximum Enclosed Area

You are given a string of fixed length \( l \) with one end fastened at the origin, and you are to place the string in the \( xy \)-plane with its other end on the \( x \) axis in such a way as to enclose the maximum area between the string and the \( x \)-axis. Show that the required shape is a semicircle.

GUIDANCE #1 — E-L SETUP: The area enclosed is of course \( A = \int y \, dx \) but this form of integral cannot be subjected to the Euler-Lagrange equations because the endpoints in \( x \) are not fixed! Remember: the E-L equations only apply when the integral we are trying to extremize has fixed endpoints in both the independent variable “\( t \)” and the dependent coordinates “\( q_i \)”. The quantities that are fixed are the length \( l \) of the string and the starting and ending \( y \)-coordinates. We must thus change variables from \((x, y)\) to \((s, y)\) where \( s \) is cumulative distance along the path. This \( s \) runs from 0 to \( l \), and \( y \) runs from 0 back to 0, so we now have fixed endpoints in both coordinates, good! Depending on which coordinate you choose as your independent variable, the integral to maximize will thus have the form \( \int_0^l L(y, \dot{y}, s) \, ds \) or \( \int_0^0 L(s, \dot{s}, y) \, dy \). (If you are bothered by the 0-to-0 limits
on that second form, see the footnote\(^1\). All you must do is translate \(dx\) from the original area integral into a form involving \(ds\) and \(dy\). Relating \(dx\), \(dy\), and \(ds\) to each other is easy once you realize that \(\text{"}ds\text{"}\) is the little step you take along your path when you change position by \(dx\) and \(dy\) ... which is exactly the line element of \(x,y\)-space: \(ds = dl \equiv |dl|\). Once you have recast your area integral in terms of \(y\) and \(s\), you can write down the E-L equation for the maximal-area path \(y(s)\) (or \(s(y)\))

**GUIDANCE #2 — E-L SOLUTION:** After you have obtained the E-L equation for \(y(s)\), you have to solve it. The simplest route is probably to use the information we were given: that the solution is a semicircle. Guessing the solution form in advance is, after-all, one of the primary techniques for solving ODEs: "Guess and Plug"! ☺ To prove that a semicircle is the solution — which is your task — you must first figure out what \(y(s)\) is for a semicircle of fixed circumference \(l\). Hint: introduce a temporary angle, express \(s\) and \(y\) in terms of it, then get rid of it. Once you’ve figured out \(y(s)\) for the semicircle, check explicitly that it satisfies your Euler-Lagrange equations. If it does, your proof is complete. ☺ But read a little further ...

**WARNING:** you must NEVER use PRIOR KNOWLEDGE or intuition about what the final path will look like when SETTING UP the integral to extremize! When you set up \(A = \int y\ dx = \int L(y,\dot{y},s) \ ds\), you must not use any relations that restrict \(y(s)\) to a semi-circle, or any other shape → \(y(s)\) must be completely free to vary in any way it wants to between the fixed endpoints. If not, you have completely destroyed the entire machinery that leads to the Euler-Lagrange equations. If this point is not 100% clear, please ask!

**Problem 4: Geodesic on a Cone**

(a) Consider a conical surface whose apex is at the origin, whose axis of symmetry runs along the +z axis, and whose half-angle is \(\alpha\). Using cylindrical coordinates, calculate the path \(s(\phi)\) of minimum distance between the endpoints \((s, \phi) = (s_0, -\pi/2)\) and \((s_0, +\pi/2)\) where \(s_0\) is a positive constant.

(b - not for points: optional part that is pretty cool ☺) Imagine that you constructed this conical surface from a rolled-up piece of paper. Let’s return the paper to its original flat shape: mentally slice the paper cone with scissors along the line \(\phi = \pi\), then unwrap the paper and flatten it on your desk. The resulting shape will look like Pac-Man. ☺ Figure out the transformation that maps the cone coordinates \(s\) and \(\phi\) onto the 2D-polar coordinates \(r\) and \(\theta\) on your piece of paper. (Hint: the angle \(\phi\) has to be rescaled.) Finally, express your geodesic in terms of the paper coordinates \(r\) and \(\theta\) and show that it is actually a straight line on the paper! ☺

**Problem 5: Cost-Effective Flying**

An airplane flies in the \((x,z)\)-plane, with \(z=0\) being ground level and +z pointing upward. The plane flies from \((x,z) = (-a,0)\) to \((+a, 0)\). The density of the air decreases with altitude, so fuel usage is reduced at higher \(z\) ... but of course, going to higher \(z\) also increases the total length of the plane’s path. What a perfect situation for mathematical optimization! Given that the cost of flying the plane is \(e^{-kz}\) per unit distance traveled along the plane’s path, find the flight path \(z(x)\) that minimizes the total cost of the flight. (Assume that the constant \(k\) is positive and that \(ka < \pi/2\) ... and remember to review the Mathematica Oddities Appendix from Homework 2.)

\(^1\) If the choice \(S = \int_0^l L(s,\dot{s}, y)\ dy\) bothers you because the endpoints in \(y\) are the same, realize that you never actually do this integral. This is important: the values of the endpoints make no appearance whatsoever in the setup of the Euler-Lagrange equations. All that matters is that the endpoints are fixed, otherwise you are solving a completely different sort of problem that is outside the applicability of the E-L equations. The E-L equations are the ODEs that specify the complete set of paths that extremize \(S\) between any fixed endpoints. Once you have obtained the E-L ODEs and found their general solution, then you can apply any specific endpoints you have, as boundary conditions, to obtain the specific path that passes through those endpoints. If you now go back and perform that integral \(S\) over a path that runs from \(y=0\) to \(0\), you would certainly manipulate it (e.g. split it into two parts) to avoid the trivial 0 answer ... but that is just a technical issue concerning a particular integral over a particular path.