The resistance exerted by fluids on moving objects is well described by a linear drag force that’s proportional to the object’s speed plus a quadratic drag force that’s proportional to the object’s speed-squared:

\[ \vec{f}_{\text{air}} = -(bv + cv^2)\hat{v} = \vec{f}_{\text{lin}} + \vec{f}_{\text{quad}} \]

(Note that this expression is only valid as long as the object moving through the air is travelling at speeds well below the speed of sound.) For spherical objects of diameter \( D \) the constants \( b \) and \( c \) have the forms \( b = \beta D \) and \( c = \gamma D^2 \), where \( \beta \) and \( \gamma \) are properties of the medium through which the object is travelling. When that medium is air at STP\(^1\) we get these specific values:

- Linear constant: \( b = \beta D \) where \( \beta_{\text{air}} = 1.6 \times 10^{-4} \text{ N} \cdot \text{s/m}^2 \)
- Quadratic constant: \( c = \gamma D^2 \) where \( \gamma_{\text{air}} = 0.25 \text{ N} \cdot \text{s}^2/\text{m}^4 \)

Let’s use these relations to develop some intuition about air resistance in everyday situations!

**Q1. Baseballs and Beach Balls**

(a) Consider a ball of diameter \( D \) that flies through the air at STP. Show that the ratio of the quadratic to the linear drag force is \( f_{\text{quad}} / f_{\text{lin}} = \left(1.6 \times 10^3 \text{ s/m}^2\right) D \).  

(b) Taking the case of a baseball, which has regulation size \( D = 7 \text{ cm} \), find the approximate “threshold” speed \( v_{\text{thresh}} \) of the baseball at which the two drag forces are equally important. Since baseball speeds are usually given in miles per hour (mph), you might want to use that unit to get a feel for your results → use the convenient approximate relation \( 1 \text{ m/s} \approx 2 \text{ mph} \) to convert from metric. Use your finding to address these questions:

- For what approximate range of speeds is it safe to treat the drag force as purely quadratic?
- Under normal conditions is it a good approximation to ignore the linear term?

(c) Answer the same questions for a beach ball of diameter 70 cm.

**Q2. So You Know: The Reynolds Number**

The ratio of the quadratic and linear drag forces on a moving sphere turns out to be \( f_{\text{quad}} / f_{\text{lin}} = R / 48 \), where the dimensionless Reynolds number is \( R \equiv Dv/\eta \), and \( \rho \) and \( \eta \) are the mass density and viscosity of the fluid causing the drag. The Reynolds number is a common measure of the relative importance of linear and quadratic drag, so you should know about its existence. When \( R \) is very large (e.g. for big, fast objects) the quadratic force dominates; when \( R \) is very small (e.g. for small, slow objects) the linear force dominates. The factor of 48 between \( R \) and the quadratic-to-linear drag ratio is for spheres; objects of other shapes have different constants of proportionality.

(a) Find the Reynolds number for a baseball (diameter 7 cm) that is thrown through STP air (density 1.29 kg/m\(^3\) and viscosity 1.7 \( \times \) \( 10^{-5} \text{ N} \cdot \text{s/m}^2 \)) at a world-class pitching speed of 100 mph.

(b) Find the Reynolds number for a steel ball bearing (diameter 2 mm) that moves at 5 cm/s through glycerin (density 1.3 g/cm\(^3\) and viscosity 12 N·s/m\(^2\) at STP).

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\(^1\) STP stands for “Standard Temperature and Pressure” and refers to \( T = 0^\circ \text{C} \) and \( P = 100 \text{ kPa} \approx 1 \text{ atmosphere} \).

\(^2\) Q1 (b) \( v_{\text{thresh}} \approx 0.02 \text{ mph} \) → quadratic totally dominates for typical baseball speeds of 80–100 mph

(c) \( v_{\text{thresh}} \approx 0.001 \text{ m/s} \) → quadratic totally dominates for typical beach ball speeds of a few m/s

\(^3\) Q2 (a) \( R \approx 3 \times 10^5 >> 48 \) → quadratic term dominant  (b) \( R \approx 0.01 << 48 \) → linear term dominant
Q3. Particle in a Constant Magnetic Field

The other main example of a velocity-dependent force is the **Lorentz force** that a magnetic field \( \vec{B} \) exerts on a moving particle of charge \( q \) and mass \( m \): \( \vec{F} = q\vec{v} \times \vec{B} \). Consider such a particle moving in the xy-plane \((z=0)\) through a region with a constant magnetic field \( \vec{B} = B\hat{z} \) where \( B \) is a positive constant.

(a) Write down the equations of motion (EOMs) for the particle in **cylindrical coordinates** \( s \) and \( \phi \). (Our particle remains in the xy-plane, so we ignore the \( z \) coordinate.) To simplify your expressions, introduce the convenient symbol \( \beta \equiv qB/m \).

(b) You should have two equations of motion, and you can use them to find the general solution \( s(t), \phi(t) \) for the particle \( \ldots \text{in theory} \). The enormous challenge is that your two differential equations are **coupled**; each one involves both functions! There is no way to proceed with a separate-and-integrate solution unless we can **decouple** the equations \( \ldots \) and for these two, we can’t. So we must turn to our other method: guess-and-plug. Make the following guess: \( s(t) = R \), where \( R \) is a constant. Plug this guess into your two EOMs and solve the greatly-simplified EOMs that result for \( \phi(t) \).

(c) Is the solution \{ \( s(t), \phi(t) \) \} you just found the **general solution** for a particle moving through this particular B-field in the xy-plane? Why or why not?

(d) Now find the equations of motion in **Cartesian coordinates** \( x \) and \( y \). Again, you should get two coupled differential equations. To help us work with them further, note that only time-derivatives of \( x \) and \( y \) appear, never \( x \) or \( y \) on their own \( \rightarrow \) we can simplify our next steps by rewriting the EOMs in terms of the velocity components \( v_x \equiv \dot{x} \) and \( v_y \equiv \dot{y} \), since those are clearly the quantities we will be solving for first.

(e) This pair of diff eqs can be decoupled! We can obtain separated = uncoupled equations for \( v_x(t) \) and \( v_y(t) \): one equation involving only \( v_x \) and its derivatives, and another equation involving only \( v_y \) and its derivatives. See if you can figure out how to do this on your own \( \ldots \) if you’re stuck, guidance is in the footnote.

(f) Solve! Get the general solution for \( v_x(t) \) and \( v_y(t) \), and finally the general solution for \( x(t) \) and \( y(t) \). Does your general solution have the right number of free parameters?

Q4. The Magnetic Force Does No Work

Magnetic fields give moving charged particles an acceleration \( \vec{a} = (q\vec{v} \times \vec{B})/m \). Prove these useful results:

- If \( \vec{a}(t) \) is always orthogonal to \( \vec{v}(t) \) — as in the magnetic-force case — then \( |\vec{v}(t)| \) never changes;
- and the converse: if \( |\vec{v}(t)| \) is constant, then \( \vec{a}(t) \) must be orthogonal to \( \vec{v}(t) \).

Stuck? \( \ldots \) Are you sure? \( \odot \) \( \ldots \) see the file "From225-6BasicPieces.pdf" on our website, and look at pieces #0 and #5.

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4 Q3 (a) Hint 1: Remember: the formula for acceleration in cylindrical coordinates is not obvious! It can be found in your lecture notes or on our 3D calculus formula sheet. \( \ldots \) Hint 2: just as in Cartesian coordinates, \( \vec{F} = m\vec{a} \) is a vector equation, so it represents more than one independent equation \( \rightarrow \) one for each component \( (\dot{s} \text{ and } \dot{\phi}) \ldots \) EOM#1: \( s\ddot{\phi} + 2\dot{s}\dot{\phi} = -\beta \dot{s} \), EOM#2: \( \ddot{s} - s\phi' = \beta s\dot{\phi} \).

(b) \( \phi(t) = -\beta t + \phi_0 \) (c) No! Hint for the reason: how many free parameters does your solution have? (d) EOMs: \( \dot{v}_x = \beta v_y \) and \( \dot{v}_y = -\beta v_x \) (e) Decoupling: To turn the 1st EOM into a separated equation for \( v_x \) alone you must substitute in \( v_y \) from the 2nd EOM \( \ldots \) but the 2nd EOM gives you \( \dot{v}_x \) not \( v_x \)!

5 Q4 Hint: to show that speed \( v = |\vec{v}(t)| \) is constant, you might as well show that speed squared is constant \( \ldots \) and that will remove an unpleasant square root from your work. \( \odot \)