The Procedure for Multidimensional Integration $\int_{\mathbb{R}^n} F(r_i) \otimes d\mathbb{R}^n$

Definition of the essential word "**PARAMETRIZE**" as used in this context:

> Express **all quantities that vary over the integral** in terms of your **integration parameters** (**IP**s) & constants.

1. Parametrize the **Region** $\mathbb{R}^n$

a. Pick your **coordinate system** $r_i = (x,y,z)$ Cartesian, or (r,θ,ϕ) spherical, or (s,ϕ,z) cylind.

b. Pick your n **integration parameters** u_j – a.k.a. IPs or "**sweeping parameters**" – that will sweep out the region $\mathbb{R}^n$. If possible, use one or more of your chosen coordinates r_i.

c. Describe the *shape* of $\mathbb{R}$ by expressing your coordinates r_i as **functions**[1] $r_i(u_j)$ of the IPs

d. Describe the *edges* of $\mathbb{R}$ by providing **bounds** on each integration parameter u_j

2. Parametrize the **Differential** $d\mathbb{R}^n$ using your coordinate system's **Line Element** $d\vec{l}$

Method 1: Visualization

$d\mathbb{R}^n = d\vec{l} \,|\, d\vec{A} \,|\, dV$ is **how much space** (length | area | volume) **you sweep out when you increase every IP u_j by du_j.** Figure it out with a sketch and/or the line element $d\vec{l}$ of your coordinate system. This method works best when the integration parameters u_j are actual coordinates r_j.

Method 2: Formalism

Defining $d\vec{l}_u \equiv \dfrac{\partial \vec{l}}{\partial u}\, du,$

$$d\vec{l}_{\text{path}} = d\vec{l}_u$$

$$d\vec{A} = d\vec{l}_u \times d\vec{l}_v$$

$$dV = \left(d\vec{l}_u \times d\vec{l}_v\right) \cdot d\vec{l}_w$$

3. Construct the **Integral** expressing *everything* in terms of your **IPs** and **constants**

Use your **coordinate functions $r_i(u_j)$** from to express *everything that varies* in the integrand *entirely* in terms of the IPs and constants. Watch out especially for spher/cylind unit vectors!

Your integral must be **doable** = something you can type into Wolfram Integrator[2], and must **make sense** = give a result that depends only on quantities that *survive the integration*. (Example of nonsense: a final result with an IP left in it!) Proper integrals have this form:

$\mathbb{R}^1$ path integral	$\mathbb{R}^2$ surface integral	$\mathbb{R}^3$ volume integral
$\int_{u_i}^{u_f} G(u)\, du$	$\int_{v_i}^{v_f}\int_{u_i}^{u_f} G(u,v)\, du\, dv$	$\int_{w_i}^{w_f}\int_{v_i}^{v_f}\int_{u_i}^{u_f} G(u,v,w)\, du\, dv\, dw$

For **vector integrals**, you will get one such scalar integral **per component**.

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[1] What to call these functions $r_i(u_j)$? **Constraint functions** is a good name, as that's what they do: constrain the coordinates to lie on your region $\mathbb{R}$. I like the descriptive **shape functions**, but we'll go with **coordinate functions**.

[2] Free integration available online at http://integrals.wolfram.com (indefinite integrals only). The new, insanely powerful WolframAlpha can do definite integrals too → see http://wolframalpha.com/examples/Calculus.html

Tips and Tricks for Multi-D Integration

➔ Choose the <u>coordinate system</u> r_i that best matches the <u>integration region $\mathbb{R}$</u>, not the integrand.

➔ If your integral gives a <u>vector result</u>, you must split it into <u>3 separate integrals</u>, one for each component. (Why? Vectors sum by components, and integrals are just that: sums.)

➔ <u>Beware of non-Cartesian unit vectors in your integrand!</u> If $\hat{r}$, $\hat{s}$, $\hat{\theta}$, or $\hat{\phi}$ appear in your integrand and are associated with coordinates over which you're integrating, you *cannot* pull them out of the integral because they're *not constant* → transform them to fixed, Cartesian unit vectors before you integrate. The one exception is in field integrals (next point): if $\hat{r}$, $\hat{s}$, $\hat{\theta}$, or $\hat{\phi}$ are associated with the *field-point* coordinates rather than the source-point coordinates, they *are* constant over the integral and can be left alone.

➔ To change the <u>direction</u> of a path integral, <u>change the bounds, not $d\vec{l}$</u>. Do not mess with the direction of $d\vec{l}_{\text{path}}$: it is tied to your coordinate system and your parametrization of the path by the strict formula $d\vec{l}_{\text{path}} = (d\vec{l}\ /\ du)\ du$. In contrast, you are completely free to choose the order of the bounds on your IP, u. Final point: if you do *both*, you'll have done *nothing* → try it and see!

➔ If your integral gives you a <u>field</u> as a result (i.e., a function like $V(\vec{r})$ or $\vec{E}(\vec{r})$) you must be careful to distinguish between **field point** and **source point** coordinates:

> *The <u>source-point</u> coordinates $\vec{\boldsymbol{r}}_{\boldsymbol{q}}$ <u>vary</u> over the integral, while the <u>field-point</u> coordinates $\vec{\boldsymbol{r}}$ do <u>not</u> and can be treated as constants.*

Be sure to <u>label them differently</u>: use a subscript or prime to identify the source point coordinates in your expressions, and use different symbols on your sketches such as

- • for the source point (because it looks like a physical charge … to me anyway ☺)
- × for the field point (because it reminds me of a treasure map. seriously.)

➔ Always consider the **symmetries** of the system, namely <u>transformations that leave the system unchanged</u>. If a system has such symmetries, then any field it produces or quantity that describes it will *also* be unchanged under those transformations. This allows you to simplify your work in advance! For integrals producing fields, symmetries can <u>restrict the functional dependence</u> of the result. For integrals producing vectors (constant vectors or vector fields), symmetries can <u>restrict the number of components</u> you have to calculate.

➔ Be sure to <u>shift</u> and/or <u>rotate your coordinate system</u> in order to match its symmetries to those of your region of integration. Note that the *z-axis* is the axis of symmetry for both the cylindrical and spherical coordinate systems, while the *origin* is the point of symmetry for all coordinate systems. Detail to keep in mind: if you shift/rotate your coordinate system to get a nice description of $\mathbb{R}$, you must transform your *integrand too*, in the same way.