

Gauss-Green-Stokes (GGS) & Related Theorems

	Irrotational Fields	Divergenceless Fields
general form: $\int_{\mathbb{R}} df = \oint_{\partial\mathbb{R}} f$	1. $\vec{\nabla} \times \vec{F} = 0$ everywhere	1. $\vec{\nabla} \cdot \vec{B} = 0$ everywhere
Grad $\int_{\vec{a}}^{\vec{b}} \vec{\nabla} V \cdot d\vec{l} = V(\vec{b}) - V(\vec{a})$	2. $\oint \vec{F} \cdot d\vec{l} = 0 \quad \forall$ closed loop	2. $\oint \vec{B} \cdot d\vec{A} = 0 \quad \forall$ closed surface
Stokes $\int_{\text{Surface}} (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} = \oint_{\partial\text{Surface}} \vec{E} \cdot d\vec{l}$	3. $\int_{\vec{a}}^{\vec{b}} \vec{F} \cdot d\vec{l}$ is path-independent	3. $\int_S \vec{B} \cdot d\vec{A}$ depends only on ∂S
Gauss $\int_{\text{Volume}} (\vec{\nabla} \cdot \vec{E}) dV = \oint_{\partial\text{Volume}} \vec{E} \cdot d\vec{A}$	4. $\vec{F} = \vec{\nabla} g$ for some $g(\vec{r})$	4. $\vec{B} = \vec{\nabla} \times \vec{A}$ for some $\vec{A}(\vec{r})$

Vector-Calculus Identities

Triple Products: $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \quad \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

1st Deriv: $\vec{\nabla}(fg) = f(\vec{\nabla}g) + g(\vec{\nabla}f) \quad \vec{\nabla}(\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{A}$
 $\vec{\nabla} \cdot (f\vec{A}) = f(\vec{\nabla} \cdot \vec{A}) + \vec{A}(\vec{\nabla}f) \quad \vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$
 $\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) - \vec{A} \times \vec{\nabla}f \quad \vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$

2nd Deriv: $\vec{\nabla} \times (\vec{\nabla}f) = 0 \quad \vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0 \quad \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

Cartesian Coordinates

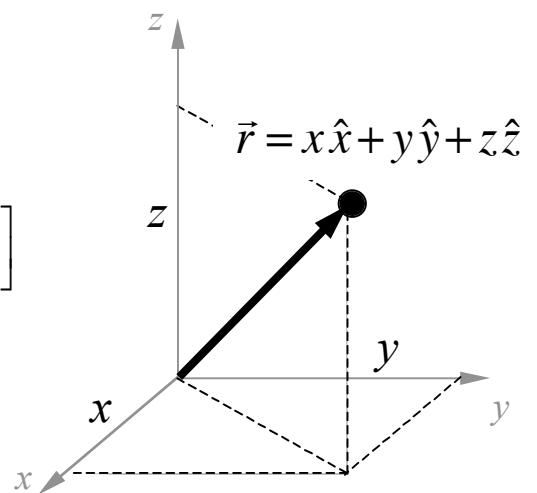
Line Element: $d\vec{l} = dx \hat{x} + dy \hat{y} + dz \hat{z}$

Gradient: $\vec{\nabla}V = \frac{\partial V}{\partial x} \hat{x} + \frac{\partial V}{\partial y} \hat{y} + \frac{\partial V}{\partial z} \hat{z}$

Divergence: $\vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$

Curl: $\vec{\nabla} \times \vec{E} = \hat{x} \left[\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right] + \hat{y} \left[\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right] + \hat{z} \left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right]$

Laplacian: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$



Spherical Coordinates

Line Element: $d\vec{l} = dr \hat{r} + r d\theta \hat{\theta} + r \sin\theta d\phi \hat{\phi}$

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \tan^{-1}(\sqrt{x^2 + y^2} / z)$$

$$\phi = \tan^{-1}(y / x)$$

$$\hat{x} = \sin\theta \cos\phi \hat{r} + \cos\theta \cos\phi \hat{\theta} - \sin\phi \hat{\phi}$$

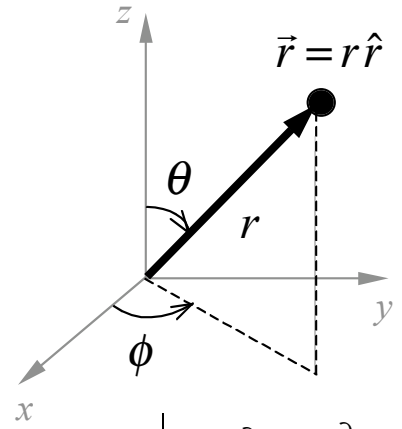
$$\hat{y} = \sin\theta \sin\phi \hat{r} + \cos\theta \sin\phi \hat{\theta} + \cos\phi \hat{\phi}$$

$$\hat{z} = \cos\theta \hat{r} - \sin\theta \hat{\theta}$$

$$\hat{r} = \sin\theta \cos\phi \hat{x} + \sin\theta \sin\phi \hat{y} + \cos\theta \hat{z}$$

$$\hat{\theta} = \cos\theta \cos\phi \hat{x} + \cos\theta \sin\phi \hat{y} - \sin\theta \hat{z}$$

$$\hat{\phi} = -\sin\phi \hat{x} + \cos\phi \hat{y}$$



Gradient: $\vec{\nabla}V = \frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} + \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi} \hat{\phi}$

Laplacian: $\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 V}{\partial \phi^2}$

Divergence: $\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} (\sin\theta E_\theta) + \frac{1}{r \sin\theta} \frac{\partial E_\phi}{\partial \phi}$

Curl: $\vec{\nabla} \times \vec{E} = \frac{\hat{r}}{r \sin\theta} \left[\frac{\partial}{\partial \theta} (\sin\theta E_\phi) - \frac{\partial E_\theta}{\partial \phi} \right] + \frac{\hat{\theta}}{r} \left[\frac{1}{\sin\theta} \frac{\partial E_r}{\partial \phi} - \frac{\partial}{\partial r} (r E_\phi) \right] + \frac{\hat{\phi}}{r} \left[\frac{\partial}{\partial r} (r E_\theta) - \frac{\partial E_r}{\partial \theta} \right]$

	∂_r	∂_θ	∂_ϕ
\hat{r}	0	$\hat{\theta}$	$\sin\theta \hat{\phi}$
$\hat{\theta}$	0	$-\hat{r}$	$\cos\theta \hat{\phi}$
$\hat{\phi}$	0	0	$-\sin\theta \hat{r}$ $-\cos\theta \hat{\theta}$

Cylindrical Coordinates

Line Element: $d\vec{l} = ds \hat{s} + s d\phi \hat{\phi} + dz \hat{z}$

$$x = s \cos\phi$$

$$y = s \sin\phi$$

$$z = z$$

$$s = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1}(y / x)$$

$$z = z$$

$$\hat{x} = \cos\phi \hat{s} - \sin\phi \hat{\phi}$$

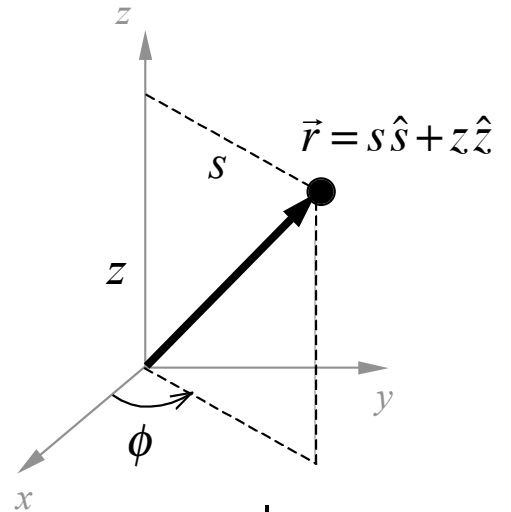
$$\hat{y} = \sin\phi \hat{s} + \cos\phi \hat{\phi}$$

$$\hat{z} = \hat{z}$$

$$\hat{s} = \cos\phi \hat{x} + \sin\phi \hat{y}$$

$$\hat{\phi} = -\sin\phi \hat{x} + \cos\phi \hat{y}$$

$$\hat{z} = \hat{z}$$



Gradient: $\vec{\nabla}V = \frac{\partial V}{\partial s} \hat{s} + \frac{1}{s} \frac{\partial V}{\partial \phi} \hat{\phi} + \frac{\partial V}{\partial z} \hat{z}$

Laplacian: $\nabla^2 V = \frac{1}{s} \frac{\partial}{\partial s} \left(s \frac{\partial V}{\partial s} \right) + \frac{1}{s^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$

Divergence: $\vec{\nabla} \cdot \vec{E} = \frac{1}{s} \frac{\partial}{\partial s} (s E_s) + \frac{1}{s} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z}$

Curl: $\vec{\nabla} \times \vec{E} = \left[\frac{1}{s} \frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} \right] \hat{s} + \left[\frac{\partial E_s}{\partial z} - \frac{\partial E_z}{\partial s} \right] \hat{\phi} + \frac{1}{s} \left[\frac{\partial}{\partial s} (s E_\phi) - \frac{\partial E_s}{\partial \phi} \right] \hat{z}$

	∂_s	∂_ϕ	∂_z
\hat{s}	0	$\hat{\phi}$	0
$\hat{\phi}$	0	$-\hat{s}$	0
\hat{z}	0	0	0