The Acoustical Physics of a Standing Wave Tube

A typical cylindrical-shaped standing wave tube (SWT) \textit{aka} impedance tube\ of length $L$ and diameter $D \ll L$ with infinitely rigid walls and closed ends is shown in the figure below:

![Standing Wave Tube (SWT)](SWT_diagram.png)

Sound energy is input to the SWT at the position $z = 0$, e.g., using a sine-wave function generator connected to some kind of acoustical transducer, such as a wafer-thin piezo-electric transducer (or a loudspeaker). Ideally-speaking, the transducer should have no frequency-dependent phase-shift(s) relative to the driving sine-wave function generator. However, in the real world, such devices do not exist. At frequencies below the lowest cutoff frequency of the SWT ($f_c \approx \frac{v}{2L} \approx \frac{2 \times 345 \text{ m/s}}{2 \times 6 \text{ cm}} \approx 111.84 \text{ Hz}$), only $1$-D type plane waves can propagate in the SWT.

Pressure ($p$) and differential/particle velocity ($u_z$) microphones are co-located at the “generic” position $z$ along the symmetry axis of the SWT. They are used to record the complex instantaneous total pressure and the instantaneous complex 1-D longitudinal/z-component of the total particle velocity at that location associated with the presence of right- and left-moving acoustic traveling plane waves propagating in the SWT. The resultant instantaneous complex pressure standing wave at the point $z$ is thus a linear superposition of these two traveling plane waves:

$$
\hat{p}(z,t) = \tilde{A}(\tilde{k}^*) e^{-i\tilde{k}^* z} e^{i\omega t} + \tilde{B}(\tilde{k}) e^{-i\tilde{k} z} e^{i\omega t} = \tilde{A}(k^*) e^{-i k^* z} e^{i\omega t} + \tilde{B}(k) e^{-i k z} e^{i\omega t} = \left[ \tilde{A}(k^*) e^{-i k^* z} + \tilde{B}(k) e^{i k z} \right] e^{i\omega t}
$$

where the complex, frequency-dependent wavenumber $\tilde{k}(\omega) \equiv k(\omega) + ik(\omega) = \omega/v(\omega) + ik(\omega)$; the * denotes complex conjugation, i.e. $\tilde{k}^*(\omega) \equiv k(\omega) - i k(\omega)$ and $i \equiv -\sqrt{-1}$. The use of $\tilde{k}^*$ ensures that we are always appropriately mathematically describing decaying exponential attenuation phenomena, i.e. for $z > 0$, using $e^{-\kappa z}$ for both right- and left-traveling waves (as opposed to unphysical, exponentially growing phenomena, i.e. $e^{i\kappa z}$ with distance).

An \{extremely\} important micro-detail here is that in order to be able to correctly compare theoretical prediction(s) to experimental data, the choice of using $e^{i\omega t}$ vs. $e^{-i\omega t}$ in the theory is in fact not arbitrary. In the UIUC Physics 199POM/498POM SWT experiment, in order to obtain the necessary phase-sensitive information on the complex nature of pressure ($p$) and 1-D particle velocity ($u_z$) as a function of frequency, the electrical signals output from the pressure and particle velocity microphone preamplifiers are each input to separate lock-in amplifiers (SRS model # DSP-830) which also use the signal output from the sine-wave function generator as the reference signal for each of the lock-in amplifiers. In our SWT experiment we have explicitly selected $0^\circ$ referencing...
of the two lock-in amplifiers to the function generator’s sine-wave signal, and thus because of the way a lock-in amplifier works, also implicitly means that we have selected the \(e^{i\omega t}\) sign convention. Had we instead selected e.g. \(180^\circ\) referencing of the lock-in amplifier to the sine-wave reference signal, then we would have implicitly instead selected the \(e^{-i\omega t}\) sign convention. Because of our \(e^{i\omega t}\) choice in referencing of the lock-in amplifiers to the function generator’s sine-wave signal, both the instantaneous complex pressure and instantaneous complex particle velocity precess (i.e. rotate) counter-clockwise (CCW) in the complex plane as time increases, since \(e^{i\omega t} = \cos \omega t + i \sin \omega t\) whereas \(e^{-i\omega t} = \cos \omega t - i \sin \omega t\).

Generically, the instantaneous complex pressure (SI units: Pascals) and instantaneous complex particle velocity (SI units: \(m/s\)) can be written as:

\[
\vec{p}(z,t) \equiv p_r(z,t) + ip_i(z,t) = |\vec{p}(z)|e^{ip}e^{i\omega t} \quad \text{and} \quad \vec{u}_z(z,t) \equiv u_r(z,t) + iu_i(z,t) = |\vec{u}_z(z)|e^{iu}e^{i\omega t}
\]

The real parts of the complex instantaneous pressure \(\vec{p}(z,t)\) and/or complex instantaneous 1-D particle velocity \(\vec{u}_z(z,t)\) are in-phase (if \(+ve\)) or \(180^\circ\) out-of phase (if \(-ve\)) relative to the reference signal output from the sine-wave function generator; the imaginary parts of the complex instantaneous pressure \(\vec{p}(z,t)\) and/or complex instantaneous 1-D particle velocity \(\vec{u}_z(z,t)\) are \(+90^\circ\) out-of-phase (if \(+ve\)) or \(-90^\circ\) out-of phase (if \(-ve\)) relative to the reference signal output from the sine-wave function generator, as shown below {for a general case/generic situation} in the so-called phase diagram – i.e. the complex plane, at time \(t = 0\):

For small amplitudes, the instantaneous complex pressure and instantaneous 3-D complex vector particle velocity are related to each other via Euler’s equation for compressible, inviscid fluid flow (inviscid fluid flow means that any/all viscous/dissipative forces \(\ll\) inertial forces):

\[
-\rho_o \frac{\partial \vec{u}(\vec{r},t)}{\partial t} = \nabla \vec{p}(\vec{r},t)
\]

where \(\rho_o = \text{mass volume density of the fluid}\) and \(\vec{u}(\vec{r},t) \cdot \nabla \{\vec{u}(\vec{r},t)\} = 0\) is assumed. For {bone-dry} air at NTP, \(\rho_o = 1.204 kg/m^3\). For the SWT with 1-D longitudinal particle velocity measurement, the corresponding 1-D Euler’s equation for plane waves propagating in the SWT reduces to

\[
-\rho_o \frac{\partial u_z(z,t)}{\partial t} = \frac{\partial p(z,t)}{\partial z}.
\]

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The total instantaneous complex pressure at the position \( z \) associated with the presence of a standing plane wave in the SWT is the instantaneous linear superposition of an (overall) right-propagating complex traveling plane wave and an (overall) left-propagating traveling plane wave:

\[
\vec{p}(z,t) = \left[ \tilde{A}(\tilde{k}) e^{-ik z} + \tilde{B}(\tilde{k}) e^{ik z} \right] e^{i\omega t}.
\]

The overall instantaneous complex pressure amplitude is in fact a linear superposition of an infinite number of individual right- and left-moving complex traveling pressure waves with complex amplitudes \( \tilde{a}_n(\tilde{k}) \) and \( \tilde{b}_n(\tilde{k}) \), \( n = 0, 1, 2, 3, \ldots \infty \) respectively, each of which are associated with the sine-wave signal output from the acoustical transducer (located at \( z = 0 \)) at times earlier than \( t = 0 \). Thus, mathematically the complex amplitudes associated with right- and left-moving complex amplitudes can each be represented by the infinite series \( \tilde{A}(\tilde{k}) = \sum_{n=0}^{\infty} \tilde{a}_n(\tilde{k}) \) and \( \tilde{B}(\tilde{k}) = \sum_{n=0}^{\infty} \tilde{b}_n(\tilde{k}) \).

Both of these series representations can be represented graphically via phasor diagrams in the complex plane, e.g. for \( \tilde{A}(\tilde{k}) = \sum_{n=0}^{\infty} \tilde{a}_n(\tilde{k}) \) as shown in the figure below for \( t = 0 \):

Precisely on a resonance of the SWT, the individual complex amplitudes \( \tilde{a}_n(\tilde{k}) \) and \( \tilde{b}_n(\tilde{k}) \) associated with the individual right- and left-moving traveling waves respectively, are perfectly in phase with each other, i.e. all of the individual relative phases \( \delta_{\tilde{a}_n} = 2n\pi = 0 \), \( \delta_{\tilde{b}_n} = 2n\pi = 0 \) and thus the overall phases \( \Delta^*_A = \sum_{n=0}^{\infty} \delta_{\tilde{a}_n} = 0 \) and \( \Delta^*_B = \sum_{n=0}^{\infty} \delta_{\tilde{b}_n} = 0 \), graphically corresponding to “straight-line” phasor diagrams for \( \tilde{A}(\tilde{k}) = \sum_{n=0}^{\infty} \tilde{a}_n(\tilde{k}) \) and \( \tilde{B}(\tilde{k}) = \sum_{n=0}^{\infty} \tilde{b}_n(\tilde{k}) \).
Explicitly writing out the complex pressure amplitudes $\tilde{A}(\hat{k}) = \sum_{n=0}^{\infty} \tilde{a}_n(\hat{k})$ and $\tilde{B}(\hat{k}) = \sum_{n=0}^{\infty} \tilde{b}_n(\hat{k})$

associated with the overall right- and left-moving complex plane waves:

$$\tilde{A}(\hat{k}) = A_0 e^{-i\delta_0} + A_0 e^{-i\delta_1} + A_0 e^{-i\delta_2} + A_0 e^{-i\delta_3} + \ldots = A_0 \left[ e^{-i\delta_0} + e^{-i\delta_1} + e^{-i\delta_2} + e^{-i\delta_3} + \ldots \right] = A_0 \sum_{n=0}^{\infty} e^{-i\delta_n}$$

And:

$$\tilde{B}(\hat{k}) = A_0 e^{-i\delta_0} + A_0 e^{-i\delta_1} + A_0 e^{-i\delta_2} + A_0 e^{-i\delta_3} + \ldots = A_0 \left[ e^{-i\delta_0} + e^{-i\delta_1} + e^{-i\delta_2} + e^{-i\delta_3} + \ldots \right] = A_0 \sum_{n=0}^{\infty} e^{-i\delta_n} = \tilde{A}(\hat{k})$$

The multiplicative phase factor $e^{-i\delta_n}$ associated with the $n^{th}$ term in each of the two infinite series arises from the fact that each such contributing wave had to originate at an earlier time, $t_n < 0$ in order for all such waves to arrive simultaneously at the $z = z$ position at the time $t = t$. Note that since $e^{i\omega t}$ rotates complex quantities CCW in the complex plane as the time $t$ increases, the sign in the argument of the $e^{-i\delta_n}$ phase factor associated with waves arriving at the $z = z$ position at the time $t = t$ from the earlier time $t_n < 0$ must be negative. Since the elapsed time for $n$ round trips of right- or left-moving waves propagating in the SWT is $\Delta t_n \equiv t - t_n = n(2L)/v = 2nL/v$, then the complex phase shift associated with $n$ round trips of right- or left-moving waves propagating in the SWT is $\delta_{tn} = 2\omega\Delta t_n = 2nL\omega/\tilde{v} = 2nk\lambda L$ and thus $e^{-i\delta_n} = e^{-2i\omega t_n L} = e^{2i\omega(2nL/v)} = e^{-2i\omega(2nL/v) - i\pi}$.

Thus: $\tilde{A}(\hat{k}) = A_0 + A_0 e^{-2kL} e^{-2ikL} + A_0 e^{-4kL} e^{-4ikL} + A_0 e^{-6kL} e^{-6ikL} + \ldots = A_0 \sum_{n=0}^{\infty} e^{-2nk\lambda L} e^{-2i\delta_n} = \tilde{B}(\hat{k})$

Note that since the end walls of the SWT (located at $z = 0$ and $z = L$ respectively) are assumed to be infinitely rigid, we have tacitly/implicitly assumed that no additional phase shift(s) of the right-/left-moving traveling waves occurs upon reflection at the end walls. If such reflection-induced phase shifts were to occur, then additional phase factors $e^{-i\phi_0}$ and $e^{-i\phi_L}$ would need to be included in the above expressions in order to explicitly take into account/properly mathematically describe general/generic phase shifts associated with reflection of the individual right- and left-moving plane waves at {non-perfectly rigid} end-walls of the SWT.

Is it possible to obtain an analytic, closed-form expression for the infinite series associated with the complex amplitudes $\tilde{A}(\hat{k})$ and $\tilde{B}(\hat{k})$? The answer is a most definite yes!

Defining $t \equiv 2\kappa L > 0$ and $x \equiv 2kL$, and noting that $\sum_{n=0}^{\infty} e^{-nt} e^{-inx} = \sum_{n=0}^{\infty} e^{-nt} \cos nx - i \sum_{n=0}^{\infty} e^{-nt} \sin nx$ the analytic/closed-form expressions for the two $\infty$ series on the RHS of this relation, for $t > 0$ are [11]:

$$\sum_{n=0}^{\infty} e^{-nt} \sin nx = \frac{1}{2} \left( \frac{\sin x}{\cosh t - \cos x} \right)$$

and:

$$\sum_{n=0}^{\infty} e^{-nt} \cos nx = \frac{1}{2} \left( \frac{\sinh t}{\cosh t - \cos x} + 1 \right)$$
Thus:  
\[ \sum_{n=0}^{\infty} e^{-nt} \cos nx - i \sum_{n=0}^{\infty} e^{-nt} \sin nx = \sum_{n=0}^{\infty} e^{-nt} e^{-inx} = \frac{1}{2} \left( \frac{\sinh t}{\cosh t - \cos x} + 1 \right) - i \left( \frac{\sin x}{\cosh t - \cos x} \right) \]

Hence the analytic/closed-form expression for complex \( \tilde{A}(k) = \tilde{B}(k) \) is:

\[ \tilde{A}(k) = \tilde{B}(k) = A_o \left[ \sum_{n=0}^{\infty} e^{-2knL} e^{-2in\omega L} \right] = \frac{1}{2} \left[ 1 + \frac{\sinh (2\kappa L)}{\cosh (2\kappa L) - \cos (2kL)} \right] - \frac{1}{2} \left[ \frac{\sin (2kL)}{\cosh (2\kappa L) - \cos (2kL)} \right] \]

Thus, the overall instantaneous complex pressure amplitude \( \tilde{p}(z,t) \) is:

\[ \tilde{p}(z,t) = \left[ \tilde{A}(k) e^{-ikz} + \tilde{B}(k) e^{ikz} \right] e^{i\omega t} = \tilde{A}(k) \left[ e^{-ikz} + e^{ikz} \right] e^{i\omega t} = 2\tilde{A}(k) e^{-\kappa z} \cos(kz) e^{i\omega t} \]

Defining:  
\[ \tilde{\varphi}(k) \equiv \left[ 1 + \frac{\sinh (2\kappa L)}{\cosh (2\kappa L) - \cos (2kL)} \right] - \frac{1}{2} \left[ \frac{\sin (2kL)}{\cosh (2\kappa L) - \cos (2kL)} \right] \]

Then \( \tilde{p}(z,t) \) can be written as:  
\[ \tilde{p}(z,t) = A_o \tilde{\varphi}(k) e^{-\kappa z} \cos(kz) e^{i\omega t} \]

The 1-D complex particle velocity is of the general form \( \tilde{u}_z(z,t) = \tilde{u}_o(z) e^{i\omega t} \) and is related to the complex pressure \( \tilde{p}(z,t) \) via the 1-D Euler equation:

\[ -\rho_o \frac{\partial \tilde{u}_z(z,t)}{\partial t} = \frac{\partial \tilde{p}(z,t)}{\partial z} \]

Thus:

\[ \tilde{u}_z(z,t) = -i \frac{1}{\omega \rho_o} A_o \tilde{\varphi}(k) e^{-\kappa z} \left[ \kappa \cos kz + k \sin kz \right] e^{i\omega t} \]

Since \( v(\omega) = \omega/k(\omega) \) this relation can also be written as:

\[ \tilde{u}_z(z,t) = -i \frac{1}{\rho_o v} A_o \tilde{\varphi}(k) e^{-\kappa z} \left[ \frac{\kappa}{k} \cos kz + \sin kz \right] e^{i\omega t} \]

For inviscid fluid flow, note that \( (\kappa/k) \ll 1 \) or equivalently, that \( \kappa \ll k \).
The complex longitudinal particle displacement \( \xi(z,t) \) is related to the complex longitudinal particle velocity \( \mathbf{u}_z(z,t) \) via \( \mathbf{u}_z(z,t) = \partial_z \xi(z,t) / \partial t \).

Thus:

\[
\xi(z,t) = -\frac{1}{\omega \rho_o v} A_z \tilde{F}(\tilde{k}) e^{-kz} \left[ \left( \frac{\kappa}{k} \right) \cos kz + \sin kz \right] e^{i\omega t}
\]

The complex specific acoustic impedance \( \tilde{z}(z) \) of the SWT tube at the position \( z \) is defined as the ratio of complex pressure to complex longitudinal particle velocity:

\[
\tilde{z}(z) = \frac{\tilde{p}(z,t)}{\tilde{u}_z(z,t)} = \frac{\tilde{p}(z) e^{i\omega t}}{\tilde{u}_z(z) e^{i\omega t}} = \frac{\tilde{p}(z)}{\tilde{u}_z(z)} = +i \rho_o v \cos kz \left[ \left( \frac{\kappa}{k} \right) \cos kz + \sin kz \right]
\]

Note that the complex specific acoustic impedance \( \tilde{z}(z) \) of the SWT is purely imaginary, and is also a time-independent quantity, since \( \tilde{p}(z,t) \) and \( \tilde{u}_z(z,t) \) have the same time-dependence factor \( e^{i\omega t} \).

The SI units of complex specific acoustic impedance \( \tilde{z}(z) \) are \( \text{Pa} \cdot \text{s} / \text{m} \), also known simply as acoustic ohms.

The 1-D complex longitudinal acoustic intensity \( \tilde{I}_z(z) \) at the position \( z \) is defined as:

\[
\tilde{I}_z(z) = \frac{1}{2} \tilde{p}(z,t) \tilde{u}_z^*(z,t) = \frac{1}{2} \tilde{p}(z) \tilde{u}_z^*(z) = -i \frac{A_z^2}{2 \rho_o v_o} \left| \hat{F}(\tilde{k}) \right|^2 e^{-2kz} \cos(kz) \left[ \left( \frac{\kappa}{k} \right) \cos kz + \sin kz \right]
\]

The complex longitudinal acoustic intensity \( \tilde{I}_z(z) \) in the SWT is purely imaginary and is also a time-independent quantity. The SI units of complex acoustic intensity \( \tilde{I}_z(z) \) are \( \text{Watts} / \text{m}^2 \).

The time-averaged total acoustic energy density \( \langle e_{tot}(z) \rangle \) at the position \( z \) is the additive sum of the individual time-averaged acoustic potential energy density \( \langle e_{pot}(z) \rangle \) and the time-averaged acoustic kinetic energy density \( \langle e_{kin}(z) \rangle \):

\[
\langle e_{tot}(z) \rangle = \langle e_{pot}(z) \rangle + \langle e_{kin}(z) \rangle = \frac{1}{4} \left| \tilde{p}(z,t) \right|^2 + \frac{1}{4} \rho_o \left| \tilde{u}_z(z,t) \right|^2 = \frac{1}{4} \left| \tilde{p}(z) \right|^2 + \frac{1}{4} \rho_o \left| \tilde{u}_z(z) \right|^2
\]

The time-averaged energy densities are purely real, time-independent quantities. The SI units of energy density are \( \text{Joules} / \text{m}^3 \).
Precisely at one of the resonant frequencies of the SWT, $f_n = v/\lambda_n = n\pi/2L$, with both of the $p$ and $u_z$ mics located e.g. at $z = L$, then $k_nL = 2\pi L/\lambda_n = n\pi$, and hence $\cos(k_nL) = \cos(n\pi) = (-1)^n$, $\cos(2k_nL) = \cos(2n\pi) = 1$, $\sin(k_nL) = \sin(n\pi) = 0$ and $\sin(2k_nL) = \sin(2n\pi) = 0$, and thus the instantaneous overall complex pressure amplitude at $z = L$ on the $n^{th}$ resonance of the SWT becomes:

$$\tilde{P}_n(z = L, t) = (-1)^n A_0 e^{-\kappa L} \left( \frac{\sinh(2\kappa L)}{\cosh(2\kappa L) - 1} \right) e^{i\omega_n t}$$

Now since $\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$ and using the fact that $\tanh(\frac{1}{2} x) = \frac{\sinh x}{\cosh x - 1}$, then we see that $\coth(\frac{1}{2} x) = \frac{1}{\tanh(\frac{1}{2} x)} = \frac{\sinh x}{\cosh x - 1}$ and thus if $x = 2\kappa L$ we see that the above expression for $\tilde{P}_n(z = L, t)$ on the SWT resonances can equivalently be written as:

$$\tilde{P}_n(z = L, t) = (-1)^n A_0 e^{-\kappa L} \left[ \coth(\kappa L) + 1 \right] e^{i\omega_n t}$$

Thus, we see that there are pressure anti-nodes at both $z = 0$ and $z = L$ on the resonances of the SWT for infinitely rigid/closed end walls.

The instantaneous 1-D longitudinal particle velocity at $z = L$ on the resonances of the SWT is:

$$\tilde{u}_{z_1}(z = L, t) = -i(-1)^n A_0 e^{-\kappa L} \left( \frac{\kappa}{k_n} \right) \left( \frac{\sinh(2\kappa L)}{\cosh(2\kappa L) - 1} \right) e^{i\omega_n t}$$

Again, using the relation $\coth(\frac{1}{2} x) = \frac{1}{\tanh(\frac{1}{2} x)} = \frac{\sinh x}{\cosh x - 1}$ the longitudinal particle velocity at $z = L$ on one of the resonances of the SWT can be rewritten as:

$$\tilde{u}_{z_2}(z = L, t) = -i(-1)^n A_0 e^{-\kappa L} \left( \frac{\kappa}{k_n} \right) \left[ \coth(\kappa L) + 1 \right] e^{i\omega_n t}$$

Note that on a resonance of the SWT, $\tilde{u}_{z_2}(z = L, t)$ is $-90^\circ$ out-of-phase relative to $\tilde{P}_n(z = L, t)$.

The instantaneous 1-D longitudinal particle displacement at $z = L$ on a resonance of the SWT is:

$$\tilde{z}_{z_2}(z = L, t) = -\frac{(-1)^n}{\omega_n \rho_o^v} A_0 e^{-\kappa L} \left( \frac{\kappa}{k_n} \right) \left[ \left( \frac{\sinh(2\kappa L)}{\cosh(2\kappa L) - 1} \right) \right] e^{i\omega_n t}$$
Again, using the relation

\[
\coth \left( \frac{x}{2} \right) = \frac{1}{\tanh \left( \frac{x}{2} \right)} = \frac{\sinh x}{\cosh x - 1}
\]

the longitudinal particle displacement at \( z = L \) on one of the resonances of the SWT can be rewritten as:

\[
\tilde{\xi}_{z_n} (z = L, t) = -\frac{(-1)^{n}}{\omega_n \rho_o} A_0 e^{-\kappa L} \left( \frac{\kappa}{k_n} \right) \left( \coth(\kappa L) + 1 \right) e^{i\omega_n t}
\]

Thus we see that on the resonances of the SWT, \( \tilde{\xi}_{z_n} (z = L, t) \) is \(-90^\circ\) out of phase relative to \( \tilde{u}_{z_n} (z = L, t) \) and is \(-180^\circ\) out of phase relative to \( \tilde{p}_n (z = L, t) \).

In terms of a phasor diagram, the complex pressure \( \tilde{p}_n (z = L, t) \), complex longitudinal particle velocity \( \tilde{u}_{z_n} (z = L, t) \) and complex longitudinal displacement \( \tilde{\xi}_{z_n} (z = L, t) \) on the resonances of the SWT as observed at \( z = L \) and at time \( t = 0 \) are oriented as shown in the figure below:

The complex specific acoustic impedance at \( z = L \) on the resonances of the SWT is purely imaginary and time-independent:

\[
z_n (z = L) = \tilde{p}_n (z = L, t) / \tilde{u}_{z_n} (z = L, t) = +i \rho_o v \left( \frac{k_n}{\kappa} \right)
\]

The complex longitudinal intensity at \( z = L \) on the resonances of the SWT is also purely imaginary and time-independent:

\[
\tilde{I}_{z_n} (z = L) = \frac{1}{2} \tilde{p}_n (z = L) \tilde{u}_{z_n}^* (z = L) = +i \frac{A_o^2}{2 \rho_o v} e^{-2\kappa L} \left( \frac{\kappa}{k_n} \right) \left[ \coth(\kappa L) + 1 \right]^2
\]
Note that since there are acoustic standing waves present in the SWT, the complex longitudinal sound intensity \( \tilde{I}_z(z) \) must be purely reactive (i.e. purely imaginary). Any non-zero value associated with the real component of \( \tilde{I}_z(z) \) is due to an actual \{time-averaged\} flux, or flow of energy down the SWT – this cannot be due to a standing wave – only a traveling sound wave can have this!

The time-averaged total energy density at the position \( z = L \) on a resonance of the SWT is:

\[
\langle e_{\text{tot}}(z = L) \rangle \equiv \langle e_{n,\text{pot}}(z = L) \rangle + \langle e_{n,\text{kin}}(z = L) \rangle = \frac{1}{4} \left[ \frac{p_n(z = L,t)}{\rho_0 v^2} \right]^2 + \frac{1}{4} \rho_o |\tilde{\mu}_z(z = L,t)|^2 \\
= \frac{1}{4} \frac{A_o^2}{\rho_0 v^2} e^{-2\kappa L} \left[ \coth(\kappa L) + 1 \right]^2 + \frac{1}{4} \frac{A_o^2}{\rho_0 v^2} e^{-2\kappa L} \left( \frac{\kappa}{k_n} \right)^2 \left[ \coth(\kappa L) + 1 \right]^2
\]

Or:

\[
\langle e_{\text{tot}}(z = L) \rangle \equiv \langle e_{n,\text{pot}}(z = L) \rangle + \langle e_{n,\text{kin}}(z = L) \rangle = \frac{1}{4} \frac{A_o^2}{\rho_0 v^2} e^{-2\kappa L} \left[ 1 + \left( \frac{\kappa}{k_n} \right)^2 \right] \left[ \coth(\kappa L) + 1 \right]^2
\]

Note that since \( \kappa / k_n \ll 1 \) for inviscid fluid flow, we see that \( \langle e_{n,\text{pot}}(z = L) \rangle \gg \langle e_{n,\text{kin}}(z = L) \rangle \) on the resonances of the SWT.

**References:**


In this reference, please note that the expression \( \sum_{n=0}^{\infty} e^{-nt} \sin nx \left( \frac{\sin x}{\cosh t - \cos x} \right) \) is factually in error in the 2\textsuperscript{nd} and 3\textsuperscript{rd} quadrants (where \( \cos x < 0 \)). The correct expression, valid in all four quadrants is:

\[
\sum_{n=0}^{\infty} e^{-nt} \sin nx \left( \frac{\sin x}{\cosh t - \cos x} \right)
\]