Vibrations of Circular Membranes (e.g. Drums) and Circular Plates

Solutions to the wave equation in 2 dimensions (mathematically non-trivial...)
— cylindrical symmetry ⇒ so-called Bessel functions are solutions to the wave equation.

**Boundary condition:**
— circular membrane (drum head) is clamped at the edge ⇒ a displacement node at the edge

2-D wave equation has Bessel function solutions in the radial ($r$) direction: $J_m(x_{mn}) = J_m(k_{mn}r)$,
$x_{mn} = k_{mn}r$ (dimensionless quantity), where $k_{mn} = \text{wavenumber} = 2\pi/\lambda_{mn}$, and $m, n = 0, 1, 2, 3, \ldots$
The index $m$ refers to the so-called order # of the Bessel function, and the index $n = \text{node} #.$

The boundary condition that the membrane is attached at its edge requires that there be a displacement node at $r = a = \text{radius of drum head}$ – gives rise to distinct modes of vibration of the drum head (see 3-D pix on next page):
We need two indices \((m, n)\) to uniquely specify the modal vibration harmonics on a circular membrane because it is a 2-dimensional object. The 2-D transverse displacement amplitude is e.g. 

\[
\psi_{m,n}^{\text{disp}}(r, \varphi, t) = A_{m,n} J_m(k_{m,n} r) \cos(m \varphi) \cos(\omega_{m,n} t)
\]

Figure 3.6. First 14 modes of an ideal membrane. The mode designation \((m, n)\) is given above each figure and the relative frequency below. To convert these to actual frequencies, multiply by \((2.405/2\pi)\sqrt{T/\sigma}\), where \(\sigma\) is the membrane radius.
**Example:** Frequency scan of the resonances associated with the modal vibrations of a Phattie single-head 12” custom tom drum using the UIUC Physics 193POM modal vibrations PC-based data acquisition system:

![Graph of Phattie 12” Single-Head Tom Drum - Standard Cut Bearing Edge](image)

Vibrations of Circular Plates: clamped vs. free vs. simply supported edges:

**Table 3.1.** Vibration frequencies of a circular plate with clamped edge.

<table>
<thead>
<tr>
<th>n</th>
<th>$f_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0.4694 c_L h / a^2$</td>
</tr>
<tr>
<td>2</td>
<td>$2.08 f_1$</td>
</tr>
<tr>
<td>3</td>
<td>$3.41 f_1$</td>
</tr>
<tr>
<td>4</td>
<td>$5.00 f_1$</td>
</tr>
<tr>
<td>5</td>
<td>$6.82 f_1$</td>
</tr>
<tr>
<td>6</td>
<td>$2.08 f_1$</td>
</tr>
<tr>
<td>7</td>
<td>$3.41 f_1$</td>
</tr>
<tr>
<td>8</td>
<td>$5.00 f_1$</td>
</tr>
<tr>
<td>9</td>
<td>$6.82 f_1$</td>
</tr>
</tbody>
</table>

**Table 3.2.** Vibration frequencies of a circular plate with free edge.

<table>
<thead>
<tr>
<th>n</th>
<th>$f_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1.73 f_1$</td>
</tr>
<tr>
<td>2</td>
<td>$3.91 f_1$</td>
</tr>
<tr>
<td>3</td>
<td>$6.71 f_1$</td>
</tr>
<tr>
<td>4</td>
<td>$10.07 f_1$</td>
</tr>
<tr>
<td>5</td>
<td>$13.92 f_1$</td>
</tr>
</tbody>
</table>

**Table 3.3.** Vibration frequencies of a circular plate with a simply supported edge.

<table>
<thead>
<tr>
<th>n</th>
<th>$f_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$0.2287 c_L h / a^2$</td>
</tr>
<tr>
<td>2</td>
<td>$2.80 f_1$</td>
</tr>
<tr>
<td>3</td>
<td>$5.15 f_1$</td>
</tr>
<tr>
<td>4</td>
<td>$5.98 f_1$</td>
</tr>
<tr>
<td>5</td>
<td>$9.75 f_1$</td>
</tr>
<tr>
<td>6</td>
<td>$14.09 f_1$</td>
</tr>
<tr>
<td>7</td>
<td>$14.91 f_1$</td>
</tr>
<tr>
<td>8</td>
<td>$20.66 f_1$</td>
</tr>
<tr>
<td>9</td>
<td>$26.99 f_1$</td>
</tr>
</tbody>
</table>
Chladni’s Law (1802)

\[ f_{m,n} = \nu (m + 2n)^p \]

- \( m \) and \( n \) are integers (e.g. 0, 1, 2, 3, … etc)
- For flat circular plates: exponent \( p = 2 \)
- For non-flat circular plates (e.g. cymbals): exponent \( p < 2 \)

**Figure 3.8.** Vibrational modes of circular plates: (a) free edge and (b) clamped or simply supported edge. The mode number \((n, m)\) gives the number of nodal diameters and circles, respectively.
Modal Vibrations of Cymbals (continued)

Figure 20.1. Hologram interferograms of four of the modes of vibration in a cymbal: (a) (3, 0) mode; (b) (5, 0) mode; (c) (6, 0) mode; and (d) (13, 2), a combination of two modes (Rossing and Peterson, 1982).

Figure 20.2. Modes of vibration of a 38-cm cymbal. The first six modes resemble those of a flat plate, but after that the resonances tend to be combinations of two or more modes (Rossing and Peterson, 1982).

Figure 20.3. Modal frequencies of a 60-cm cymbal (Rossing, 1982).
Modal Vibration of Flat Rectangular Plates
Stretched Rectangular Membranes

Edges of flat rectangular plate can be fixed or free, or simply supported…
⇒ different boundary conditions for wave equation on rectangular plate…
⇒ different allowed solutions for vibrational modes – again, two indices $m, n$

![Modal vibration of flat rectangular plates](image)

**Fig. 15.** Some of the modes of vibration of a stretched rectangular membrane. The length of the membrane is 1.41 times the width.

![Modal vibration of square plate](image)

**Fig. 16.** Some of the modes of vibration of a square plate fixed at the center and set into oscillation by bowing.
3. Two-Dimensional Systems: Membranes, Plates, and Shells

![Graphical representation of modes](image)

**Figure 3.12.** Graphical construction of combination modes in a square isotropic plate: (a) $(2, 0) - (0, 2)$, $x$ mode; (b) $(2, 0) + (0, 2)$, ring mode; (c) $(2, 1) - (1, 2)$ mode; and (d) $(2, 1) + (1, 2)$ mode.

![Diagram of modes](image)

**Figure 3.13.** The first 10 modes of an isotropic square plate with free edges. The modes are designated by $m$ and $n$, the numbers of nodal lines in the two directions, and the relative frequencies for a plate with $\nu = 0.3$ are given below the figures.
Vibrating Plates

Figure 3.9. Chladni patterns showing the vibrational modes of rectangular plates of different shapes: (a) $L_x/L_y = 2$; (b) $L_x/L_y = 3/2$. (Waller, 1949).

Figure 3.10. Mixing of the $(2, 0)$ and $(0, 2)$ modes in rectangular plates with different $L_x/L_y$ ratios (after Waller, 1961).
3-Dimensional Vibrations
Handbells & Church Bells

The two integers \((m, n)\) denote the number of complete nodal \((m)\) meridians extending over top of bell (= \(1/2\) of such nodes along a circumference) and \(n\) = number of nodal circles. Note that effectively, bell is vibrating as a 2-D object!
Handbells/Churchbells

\[ m = 2 \]
\[ m = 3 \]
\[ m = 4 \]
\[ m = 5 \]
\[ m = 6 \]

Group 0
(2, 0) 1.0
(3, 0) 3.0

Group I
0.61
(4, 1) 5.4
0.44
(5, 1) 7.6
0.44
(6, 1) 10.2

Group II
0.31
(2, 1) 7.4
0.30
(4, 1) 4.8

Group III
0.17
(4, 1) 5.9
0.14
(5, 1) 9.0
0.13
(6, 1) 12.6

End view

\[ E = 21.15 \] Periodic table of vibrational modes in a handbell. Below each \( g \) are the relative modal frequencies for a Malmark C5 handbell. At lower \( m, n \) gives the number of nodal meridians \( 2m \) and nodal circles \( n \) (Rossing \( rrii, 1987 \)).

Figure 21.16: Time-average hologram interferograms of vibrational modes in a C5 handbell (Rossing et al., 1984).
Vibrational Modes of a Guitar

Top surface, all by itself:
Modal Vibrations of Acoustic/Classical Guitar:

**Figure 9.9.** (a) Modes of a folk guitar top (Martin D-28) with the back and ribs in sand. (b) Modes of the back with the top and ribs in sand. (c) Modes of the air cavity with the guitar body in sand. Modal designations are given above the figures and modal frequencies below (Rossing et al., 1985).

**Figure 9.16.** Time-averaged holographic interferograms of top-plate modes of a guitar (Guitar BR11). The resonant frequencies and $Q$ values of each mode are shown below the interferograms (Richardson and Roberts, 1985).
Example: Frequency scan comparison of the mechanical resonances associated with the modal vibrations of a Martin D16 vs. a Martin 000C16 guitar using the UIUC Physics 193POM modal vibrations PC-based data acquisition system:
Modal Vibrations of Violins/Violas/Cellos, etc.

**Figure 3.18.** Chladni patterns showing two modes of vibration in the top and back of a viola (Hutchins, 1977).

**Figure 10.13.** Modes of a violin air cavity. Mode frequencies are from Roberts and Rossing (1997).

**Figure 10.12.** Interferograms of two air modes in violins using electronic TV holography. (a) $A_2$ mode excited by sound from a loudspeaker (from Saldner et al., 1996); (b) $A_1$ mode excited by applying a sound pressure internally (Roberts and Rossing, 1997).
**Figure 10.14.** Time-average holographic interferograms of a free violin top plate and back plate (Butchins et al., 1971).

**Figure 10.15.** Interferograms of the top and back plates of a violin at 100 μs, 125 μs, 250 μs, and 450 μs after application of a bridge impulse parallel to the top plate. Note the wave propagation in the top plate is outward from both bridge feet and in the back plate it is outward from the soundpost (Molin et al., 1990).
NOTE:
Some tables and figures are taken from the course text “The Acoustical Foundations of Music” by John Backus, second edition.

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