The thick line is described by
\[ x \cos \phi + y \sin \phi = R \]
Central Slice Theorem

The projection $g_\phi (x')$ can thus be calculated as a set of line integrals, each at a unique $x'$.

$$p (\phi , x')= \int_\infty^{\infty} \int_\infty^{\infty} f(x,y) \delta (x \cos \phi + y \sin \phi - x') \, dx \, dy$$

Or alternatively,

$$p (\phi , x')= \int_0^{2\pi} \int_0^{\infty} f(r, \theta) \delta (r \cos (\phi - \theta ) - x') \, r \, dr \, d\theta$$

in polar coordinates.

Again $p (\phi , x')$ can be treated as a 1D function of $x'$ at a given angle $\phi$. 

NPRE 435, Principles of Imaging with Ionizing Radiation, Fall 2022
Line Impulse Signal (1)

\[ \delta_L(x, y) = \delta(x \cos \theta + y \sin \theta - l) \]

where \( \delta(x) = \begin{cases} > 0, & x \cos \theta + y \sin \theta = l \\ 0, & \text{otherwise} \end{cases} \)
The value of the projection function $p_{\phi}(x')$ at this point is the integral of the function of $f(x,y)$ along the straight line:

$$x' = x \cos \phi + y \sin \phi$$

The integral of a line impulse function and a given 2-D signal gives the projection data from a given view ...

$$p(\phi, x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \phi + y \sin \phi - x') dx dy$$
X-ray Projection Revisited

CT projection measure line integral

\[ p(x) = \text{CT detector array output} \]

\[ p(x) = \int_{y=-\infty}^{y=\infty} f(x, y) \, dy \]
Review of 2-D Analytical Reconstruction Methods

Projection Data

Projection data $p(\phi, x')$

Incident X-rays

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \phi + y \sin \phi - x') dx dy$

Detected $p(\phi, x')$
Simple Backprojection

inverse = 1 back projection
Review of 2-D Analytical Reconstruction Methods
Back Projection Operation

Incident X-rays

\( f(x,y) \)

Detected \( p(\phi, x') \)

NPRE 435, Principles of Imaging with Ionizing Radiation, Fall 2022
Simple Backprojection

Crude Idea 1: Take each projection and smear it back along the lines of integration it was calculated over.

Result from a back projection from a single view angle:

\[ b_\phi (x,y) = \int p_\phi (x') \delta(x \cos \phi + y \sin \phi - x') \, dx' \]

Adding up all the back projections from all the angles gives,

\[ f_{\text{simple back-projection}} (x,y) = \int b_\phi (x,y) \, d\phi \]

\[ f_{\text{simple back-projection}} (x,y) = \int_0^\pi \int_{-\infty}^{\infty} d\phi \int_{-\infty}^{\infty} p_\phi (x') \delta(x \cos \phi + y \sin \phi - x') \, dx' \]
Simple Backprojection
Simple Backprojection

3 projections

4 projections

many projections

Original object
Simple Back-projection and the 1/r Blurring

Figure 11.14. (a) Image produced by back-projecting the sinogram given in figure 11.12(b); (b) the response to a point object in the middle of the field of view.

From Medical Physics and Biomedical Engineering, Brown, IoP Publishing
Impulse Response Function of Simple Backprojection Operator

\[ h_b(r) = 1/r \]

\[ f_b (x,y) = f(x,y) * 1/r \]

\[ F_b (\rho, \phi) = F (\rho, \phi) / \rho, \quad \text{since } F\{1/r\} = 1/\rho \]

Back projected image is blurred by convolution with 1/r
Central Slice Theorem

\[ F(p(\phi, x')) = F(r, \phi) \]
The nature of the 1/r blurring:
Radon transform produced equally spaced radial sampling in Fourier domain.
Simple Backprojection and Inverse Radon Transform

Crude Idea 1: Take each projection and smear it back along the lines of integration it was calculated over.

Result from a back projection from a single view angle:

\[ b_{\phi}(x,y) = \int p_{\phi}(x') \delta(x \cos \phi + y \sin \phi - x') \, dx' \]

Adding up all the back projections from all the angles gives,

\[ \hat{f}_{\text{back-projection}}(x,y) = \int b_{\phi}(x,y) \, d\phi \]

\[ \hat{f}_{\text{simple backprojection}}(x,y) = \int_{0}^{\pi} d\phi \int_{-\infty}^{\infty} p_{\phi}(x') \cdot \delta(x\cos\phi + y\sin\phi - x') \, dx' \]

\[ \hat{f}_{\text{Inverse Radon transform}}(x,y) = \int_{0}^{\pi} d\phi \int_{-\infty}^{\infty} F^{-1}\{F[p_{\phi}(x')] \cdot |w|\} \cdot \delta(x\cos\phi + y\sin\phi - x') \, dx' \]
Review of 2-D Analytical Reconstruction Methods

Projection Data

Projection data $p(\phi, x')$

Incident X-rays

Detected $p(\phi, x')$

$$p(\phi, x') = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \phi + y \sin \phi - x') dx dy$$
Filtered Backprojection (FBP), What and why?
Central Slice Theorem

Integrate intensities along x-direction

The more angles used, the better the Fourier space image is filled

Projection profiles

Create lines in central slice of entire DFT image

1-D DFTs

http://engineering.dartmouth.edu/courses/engs167/12%20Image%20reconstruction.pdf
Inverse Radon Transform

Simple backprojection

\[ \hat{f}_{\text{simple backprojection}}(x, y) = \int_{0}^{\pi} d\phi \int_{-\infty}^{\infty} p_{\phi}(x') \cdot \delta(x\cos\phi + y\sin\phi - x') dx' \]
Simple Backprojection

3 projections  
4 projections  
many projections

Original object
Inverse Radon Transform

Simple backprojection

\[ \hat{f}_{\text{simple backprojection}}(x, y) = \int_{0}^{\pi} \int_{-\infty}^{\infty} p_{\phi}(x') \cdot \delta(x\cos\phi + y\sin\phi - x') \, dx' \]

\[ \omega = \left( \nu_x^2 + \nu_y^2 \right)^{1/2} \]

(spatial frequency)
Inverse Radon Transform

Radon transform

\[^{\sim}\hat{f}(x, y) = \int_0^\pi d\phi \int_{-\infty}^{\infty} p_\phi(x') \cdot \delta(\cos \phi + \sin \phi - x') dx'

Inverse Radon transform

\[^{\sim}\hat{f}(x, y) = \int_0^\pi d\phi \int_{-\infty}^{\infty} F^{-1}\{F[p_\phi(x') \cdot |\omega|]\} \cdot \delta(\cos \phi + \sin \phi - x') dx'

\hat{f}(x, y) = \hat{f}(r, \theta) = \beta \mathcal{H}\{p_\phi(x')\}

The simple back projection operator

\[= \beta \mathcal{H}\{\mathcal{R}[f(x, y)]\}\]

\[= \mathcal{R}^{-1}\{\mathcal{R}[f(x, y)]\}\]

where

\[\mathcal{H}\{p_\phi(x')\} = \mathcal{F}_1^{-1}[|\omega|] * p_\phi(x')\]

The inverse Radon transform can be represented as a filtering process followed by a back-projection operation.
But is there something missing in this discussion?, such as

Noise in data?
Possible artifacts?
Filtered Back-projection – The Optimum Filter

Have we forgot something? What about the noise in the projection data?

Where is the noise coming from?

In reality, the true projection is

\[ \hat{p}_\phi(x') = p_\phi(x') + n_\phi(x') \]
FIGURE 3  Ideal lowpass filtering of an image. Figures in the top and bottom rows demonstrate the filtering operation in the spatial and frequency domains, respectively. The left column shows the input image and the magnitude of its Fourier transform. The middle column are the circular symmetric PSF and its transfer function, where $v = \sqrt{\alpha^2 + \beta^2}$ and $r = \sqrt{\xi^2 + \eta^2}$. Images in the right column are the output image and the magnitude of its Fourier transform.
Review of Fourier Transform and Filtering

Spectral Filtering

FIGURE 4 Ideal highpass filtering of an image. Figures in the top and bottom rows demonstrate the filtering operation in the spatial and frequency domains, respectively. The left column shows the input image and the magnitude of its Fourier transform. The middle column shows the circular symmetric PSF and its transfer function, where $v = \sqrt{x^2 + y^2}$ and $r = \sqrt{x^2 + y^2}$. Images in the right column are the output image and the magnitude of its Fourier transform.
Filtered Back-projection

Ideal filter

The Ram-Lak filter

\[ H_{RL}(\omega) = \begin{cases} 
|\omega|, & (|\omega| \leq 2\pi B) \\
0, & \text{(otherwise)}
\end{cases} \]

Ram-Lak filter
Filtered Back-projection

Figure 3-4 (a) Examples of the band-limited filter function of sampled data. Note the cyclic repetitiveness of the digital filter.
Filtered Back-Projection

\[ \Delta x = 1/(2B), \]

**Sampled version**

\[
\begin{align*}
    h_{RL}(0) &= B^2 = \frac{1}{4\Delta x^2} & \text{(if } k = 0) \\
    h_{RL}(k) &= 0 & \text{(if } k \text{ even)} \\
    h_{RL}(k) &= \frac{-4B^2}{\pi^2 k^2} = \frac{-1}{\pi^2 k^2 \Delta x^2} & \text{(if } k \text{ odd)}
\end{align*}
\]

\[
\begin{align*}
    h_{SL}(k) &= \frac{-2}{\pi^2 \Delta x^2 (4k^2 - 1)} \\
    &= \frac{-8B^2}{\pi^2 (4k^2 - 1)}
\end{align*}
\]

**Figure 3-4** (b) Spatial domain filter kernels corresponding to the filter functions shown in the Ram-Lak filter is a high-pass filter with a sharp response but results in some noise enhancement, while the Shepp-Logan and the Hamming window filters are noise-smoothed filters and therefore have better SNR.
Filtered Back-projection

The Ram-Lak filter in spatial domain

\[
    h_{RL}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H_{RL}(\omega) \exp(ix\omega) \, d\omega
\]

\[
    = \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} |\omega| \exp(ix\omega) \, d\omega
\]

\[
    = 2B^2 \text{sinc}(2\pi Bx) - B^2 \text{sinc}^2(\pi Bx)
\]

Applying the Ram-Lak filter in filtered backprojection

\[
    \hat{f}(x, y) = \frac{1}{\pi} \int_{0}^{\pi} d\phi \int_{-\infty}^{\infty} dx' \ p_{\phi}(x') h(x \cos\phi + y \sin\phi - x') \ p_{\phi}^*(x')
\]
Simple and Filtered Back-projection

\[ p_\phi(x') = \int_{-\infty}^{\infty} dx' \ p_\phi(x') h(x \cos \phi + y \sin \phi - x') \]

From Computed Tomography, Kalender, 2000.
Simple and Filtered Back-projection

FIGURE 13-28. Simple backprojection is shown on the left; only three views are illustrated, but many views are actually used in computed tomography. A profile through the circular object, derived from simple backprojection, shows a characteristic $1/r$ blurring. With filtered backprojection, the raw projection data are convolved with a convolution kernel and the resulting projection data are used in the backprojection process. When this approach is used, the profile through the circular object demonstrates the crisp edges of the cylinder, which accurately reflects the object being scanned.

Chapters 12 & 13, The Essential Physics of Medical Imaging, Bushberg
Radon Transform and Sinogram

http://tech.snmjournals.org/cgi/content-nw/full/29/1/4/F3
Filtered Back-projection

https://www.youtube.com/watch?v=ddZeLNh9aac
Filtered Back-projection

The “ringing” artifacts in reconstructed images.

In this context it is usually manifest itself as "streak artifacts". You may, for example, see this as lines radiating from the center and outwards. The term comes from electronics and is meant in the sense of a bell-ringing.
Filtered Back-projection

The sharp boundary of the Ram-Lak filter often make the spatial domain filter kernel oscillatory. It sometime introduces the “ringing” artifacts in reconstructed images.

This can be effectively resolved by the Shepp and Logan filter

\[
H_{SL}(\omega) = \begin{cases} 
|\omega| \sin \left( \frac{\omega}{4B} \right), & |\omega| < 2\pi B \\
0, & \text{otherwise}
\end{cases}
\]

\[
h_{SL}(x) = \frac{B}{\pi^2} \left\{ \frac{1 - \cos 2\pi B[(1/4B) + x]}{(1/4B) + x} + \frac{1 - \cos 2\pi B[(1/4B) - x]}{(1/4B) - x} \right\}
\]
Filtered Back-projection

- Low spatial frequency data is overweighted. Filter to compensate for this. Weighted by $1/\omega$.

- Solution - filter each projection by $|\omega|$ to account for the uneven sampling density

**Steps:**
1) Projection operation
2) Transform projection
3) Weight with $|\omega|$ 
4) Inverse transform
5) Back project
6) Add all angles
Any chance we can define an OPTIMUM filter function for FBP?
Filtered Back-projection

For FBP with noisy projection data, one can derive an optimum filter function. The FBP reconstruction has the minimum mean square error

\[ M.S.E = E \left[ (f(x, y) - \hat{f}(x, y))^2 \right] \]

In other words, there exists a filter function that can be used in the FBP reconstruction that produces the most faithful reproduction of the original image \( f(x,y) \)
Filtered Back-projection

For a situation in which we know the power spectra for both the image function $f(x,y)$ and the noise, the optimum filter function is

$$H_{opt}(\omega, \phi) = |\omega|H_W(\omega, \phi)$$

Where

$$H_W(\omega, \phi) = \frac{H^*_D(\omega, \phi)W_P(\omega, \phi)}{|H_D(\omega, \phi)|^2W_P(\omega, \phi) + W_{PN}(\omega, \phi)}$$

Remember that

$$W_P(\omega, \phi) = |\mathcal{F}[p_\phi(x')]|^2 = \{\text{real}\{\mathcal{F}[p_\phi(x')]\}\}^2 + \{\text{imag}\{\mathcal{F}[p_\phi(x')]\}\}^2$$
Fourier Transform

- In general, Fourier transform is a complex valued signal, even if \(f(x,y)\) is real valued.
- It is sometimes useful to consider the \textit{magnitude} and \textit{phase} of the Fourier transform separately.

Fourier coefficients are complex:

\[
F(u, v) = F_R(u, v) + j \cdot F_I(u, v)
\]

Magnitude:

\[
|F(u, v)| = \sqrt{F_R^2(u, v) + F_I^2(u, v)}
\]

Phase:

\[
\angle F(u, v) = \tan^{-1} \frac{F_I(u, v)}{F_R(u, v)}
\]

An alternative representation:

\[
F(u, v) = |F(u, v)| e^{j \angle F(u, v)}
\]

- The square of the magnitude \(|F(u, v)|^2\) is referred to as the \textit{power spectrum} of the original function.
Filtered Back-projection with Optimum Filter

The optimum filter function can also be written as

\[ H_W(\omega, \phi) = \frac{H_D^*(\omega, \phi)}{|H_D(\omega, \phi)|^2 + (W_{PN}(\omega, \phi)/W_P(\omega, \phi))} \]

And then

\[ H_{opt}(\omega, \phi) = |\omega|H_W(\omega, \phi) \]

\[ = \frac{|\omega|H_D^*(\omega, \phi)}{|H_D(\omega, \phi)|^2 + 1/\text{SNR}(\omega, \phi)} \]

where \([W_P(\omega, \phi)]/[W_{PN}(\omega, \phi)]\) is the signal-to-noise ratio (SNR) of the projection data at a given view angle \(\phi\).
Filtered Back-projection

Therefore the optimum estimator (the optimum reconstruction) of the original image function \( f(x,y) \) is

\[
\hat{f}(x, y) = \frac{1}{\pi} \int_{0}^{\pi} d\phi \int_{-\infty}^{\infty} dx' \, \hat{p}_\phi^*(x') h(x \cos \phi + y \sin \phi - x')
\]

where the filtered projection data is given by

\[
\mathcal{F}_1^{-1}\left[H_{\text{opt}}(\omega, \phi)\right] * \hat{p}_\phi(x') = \hat{p}_\phi^*(x').
\]
Most of current X-ray CT works in fan-beam mode ...
X-ray Computed Tomography (CT)
Filtered Back-projection in Fan-beam Mode

Why fan beam mode?
Most modern X-ray CT system use fan (or cone) beam data acquisition scheme.
Image reconstruction with fan beam mode often provide better spatial resolution with the same dimension of sampled data as the parallel case, due to the improved sampling at the central region. This is found to be important for PET, where the intrinsic limitation on spatial resolution is on the finite detector size.
Filtered Back-projection in Fan-beam Mode

Comparing parallel beam and fan beam geometries
Filtered Back-projection in Fan-beam Mode

where $\beta'$ is the angle between central line and the line passing through the reconstructed point at $(x,y)$. And $v$ is the distance between the apex of fan and the reconstruction point $p$.

\[
v = \sqrt{\left( x \cos \alpha + y \sin \alpha \right)^2 + \left( x \sin \alpha - y \cos \alpha + R_d \right)^2}
\]

\[
\beta' = \tan^{-1} \left[ \frac{x \cos \alpha + y \sin \alpha}{x \sin \alpha - y \cos \alpha + R_d} \right]
\]

\[
x' = R_d \sin \beta
\]

\[
\phi = \alpha + \beta
\]
Filtered Back-projection in Fan-beam Mode

The basic idea:
The mathematical treatment for fan beam mode is almost identical to that for parallel beam case with changing parameters!

Starting from the one-to-one relationship between the parallel beam projection space \((\phi, x')\) and fan beam projection space \((\alpha, \beta)\),

\[
x' = R_d \sin \beta \\
\phi = \alpha + \beta
\]
Filtered Back-projection in Fan-beam Mode

\[ x' = R_d \sin \beta \]
\[ \phi = \alpha + \beta \]

Where the Jacobian \(|J|\) is

\[ |J| = \left| \frac{\partial (x', \phi)}{\partial (\alpha, \beta)} \right| = R_d \cos \beta \]
Filtered Back-projection in Fan-beam Mode

Where the Jacobian $|J|$ is

$$|J| = \left| \frac{\partial (x', \phi)}{\partial (\alpha, \beta)} \right| = R_d \cos \beta$$

and the FBP in fan beam mode becomes

$$\hat{f}(x, y) = \frac{1}{2\pi} \int_{0}^{2\pi} d\alpha \int_{-\beta_m}^{\beta_m} d\beta \ p_\alpha(\beta) h\{v \sin(\beta' - \beta)\} |J|$$
Filter Function in Fan-beam Mode

Similarly, the filter function used in fan beam mode is a direct transformation from the parallel beam counterpart

\[ h(x) = \mathcal{F}^{-1}_1[|\omega|] = \mathcal{F}^{-1}_1[H(\omega)] \]

\[ x' = R_d \sin \beta \]
\[ \phi = \alpha + \beta \]

\[ h\{\nu \sin(\beta' - \beta)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \ |\omega| \exp[i\omega \nu \sin(\beta' - \beta)] \]
Filtered Back-projection in Fan-beam Mode

FBP in parallel beam case

\[
\hat{f}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dx' p_\phi(x') h(x \cos \phi + y \sin \phi - x')
\]

\[
x' = R_d \sin \beta
\]

\[
\phi = \alpha + \beta
\]

can be converted to FBP in fan beam mode by changing integration variables

\[
\hat{f}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \int_{-\beta_m}^{\beta_m} d\beta
\]

\[
\times p_\alpha(\beta) h\{x \cos(\alpha + \beta) + y \sin(\alpha + \beta) - R_d \sin \beta\}|J|
\]
Filtered Back-projection in Fan-beam Mode

Where the Jacobian $|J|$ is

$$|J| = \left| \frac{\partial (x', \phi)}{\partial (\alpha, \beta)} \right| = R_d \cos \beta$$

and the FBP in fan beam mode becomes

$$\hat{f}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha \int_{-\beta_m}^{\beta_m} d\beta \ p_\alpha(\beta) h\{v \sin(\beta' - \beta)\} |J|$$
Back-projection and Filtering Method for Reconstruction
Simple Back-projection and the 1/r Blurring

Figure 11.12. (a) An image and (b) the Radon projection or sinogram of this image. The wavy sine like patterns explain the use of the term sinogram.

From Medical Physics and Biomedical Engineering, Brown, IoP Publishing
Simple Back-projection and the 1/r Blurring Revisited

Figure 11.14. (a) Image produced by back-projecting the sinogram given in figure 11.12(b); (b) the response to a point object in the middle of the field of view.

From Medical Physics and Biomedical Engineering, Brown, IoP Publishing
Impulse Response Function of Simple Backprojection Operator Revisited

\[ h_b(r) = \frac{1}{r} \]

\[ f_b(x,y) = f(x,y) \ast \frac{1}{r} \]

\[ F_b(\rho,\phi) = \frac{F(\rho, \phi)}{\rho} \quad \text{since } F\{1/r\} = \frac{1}{\rho} \]

Back projected image is blurred by convolution with \(1/r\)
Reconstruction with Back-projection and Filtering

If we know that the consequence of simple back-projection (under certain assumptions) is to apply a $1/r$ blurring on the input image ...

\[ b(x, y) = f(x, y) \ast \ast \left( \frac{1}{r} \right) \]

Can we envisage an alternative reconstruction method that

*Back projection first*

and then

*De-convolve the $1/r$ blurring?*
Reconstruction with Back-projection and Filtering

The scheme leads to the back-projection and filtering reconstruction method

\[ \hat{f}(x, y) = \mathcal{F}_2^{-1}[\rho B(\omega_x, \omega_y)] \]

where \( \rho \) is the radial spatial frequency and

\[ B(\omega_x, \omega_y) = \mathcal{F}_2[b(x, y)] \]
Simple Backprojection --BB

Crude Idea 1: Take each projection and smear it back along the lines of integration it was calculated over.

Result from a back projection from a single view angle:

\[ b_\phi (x,y) = \int p(\phi, x') \delta (x \cos \phi + y \sin \phi - x') \, dx' \]

Adding up all the back projections from all the angles gives,

\[ f_{\text{back-projected}} (x,y) = \int b_\phi (x,y) \, d\phi \]

\[ B(x,y) = f_b (x,y) = \int_0^\pi d\phi \int_{-\infty}^{\infty} p(\phi, x') \delta (x \cos \phi + y \sin \phi - x') \, dx' \]
Reconstruction with Back-projection and Filtering

BPF reconstruction provides relatively poor images compared with FBP results because of the following two issues:

The back projection step results in an image with infinite extend. Cutting it to NxN points for filtering leads to loss of information.

The 2-D filter function discussed above has a slope discontinuity and therefore leads to ringing artifact.
Reconstruction with Back-projection and Filtering

If high quality reconstructions were to be achieved the BPF method, the following aspects have to be considered.

Although the final reconstruction is on NxN points, the back projection step should use an matrix that is at least 2Nx2N.

We would need to apply appropriate windowing to the filter function discussed above, just like in the parallel beam case...
The procedures of BPF algorithm is VERY SIMPLE!

1. Obtain the backprojected image at least up to $2N \times 2N$ if the size of the original image matrix is $N \times N$.
2. Obtain the 2-D FFT of the full $2N \times 2N$ data array.
3. Obtain the optimum filtering in 2-D in Fourier domain by multiplication of $\rho$ filter.
4. Obtain the 2-D inverse FFT and select the $N \times N$ image in the central region.
5. Normalize the image.

Review of 2-D Analytical Reconstruction Methods

Key concepts:
Mathematical framework for modeling the projection process
• Radon transform
• Central slice theorem

Characteristics of typical projection data
• Non-uniform sampling of the 2-D Fourier transform space
• Actual measurements are associated with statistical noise and detector imperfections.

Analytical reconstruction methods
• Inverse Radon transform
• Filtered back-projection (FBP)
• FBP in fan-beam geometry
• Back-projection and filtering (BPF)