NPRE 435, Fall 2019

Chapter 2: Mathematical Preliminaries

Contents

- Signals and Systems.
- Fourier Transform Basics
- Analytical Image Reconstructing Techniques
- Iterative Reconstruction Methods
- Image Quality Assessment and System Optimization.

Reference book: Chapter 2 in <<Medical Imaging Signals and Systems>>, Prince and Links, Prentice Hall, 2006.

Signals and Systems

Reading Material:

Chapter 2 in

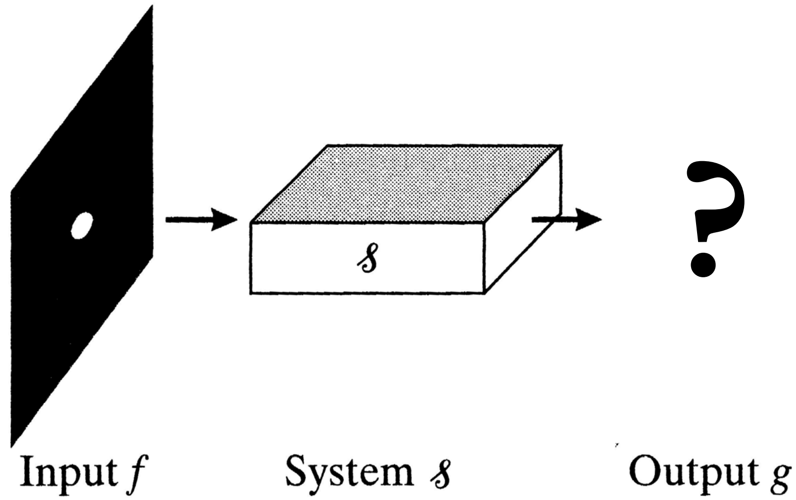
Medical Imaging Signal and Systems, 2'nd Ed.

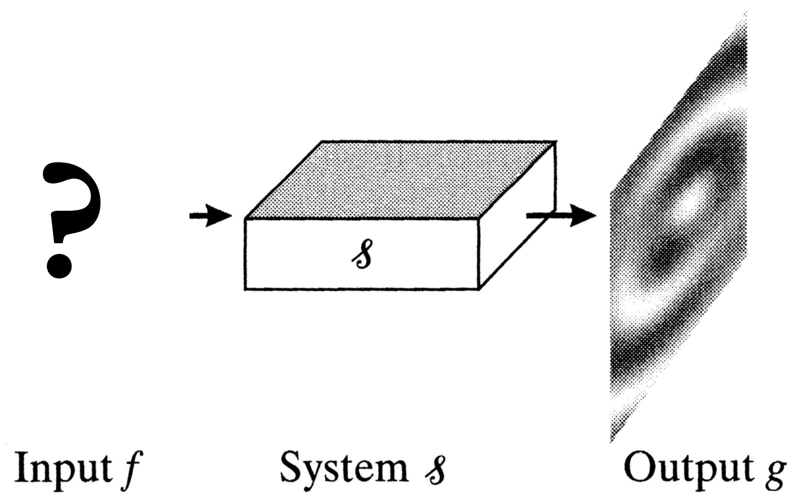
J. L. Prince et. al,

Prentice Hall, 2012.

The Basic Problems in Imaging

The forward problem: Given an input signal and the known response of a imaging system, what is the output is going to be?

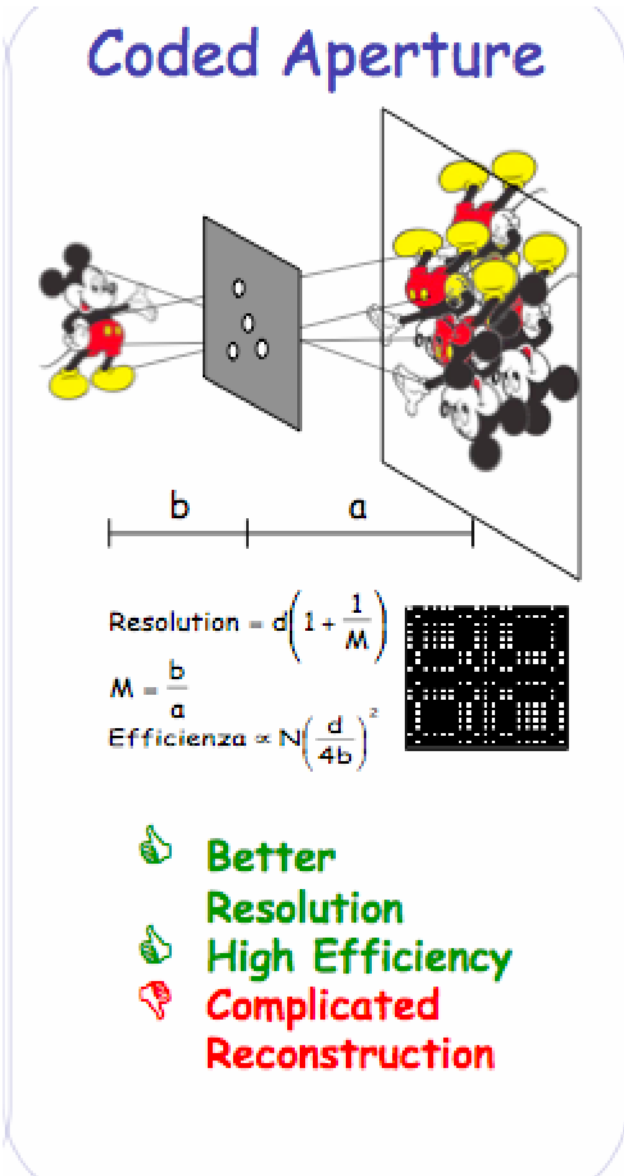




The inverse problem:

Given a output signal and the known system response, what should be the input signal that gave rise to the output data?

How to Improve the Tradeoff between Spatial Resolution and Sensitivity?



The idea of multiplexing –

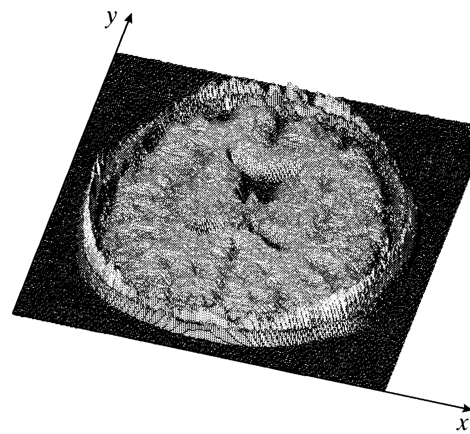
- **Each detected photon no longer corresponds to a unique emission location in the 2-D source plane.**
- **Information content per detected photon is decreased.**
- **No of detected photons is increased.**

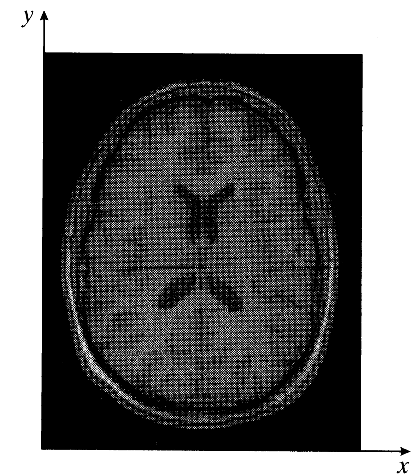
Introduction to Signals

- Continuous signal:

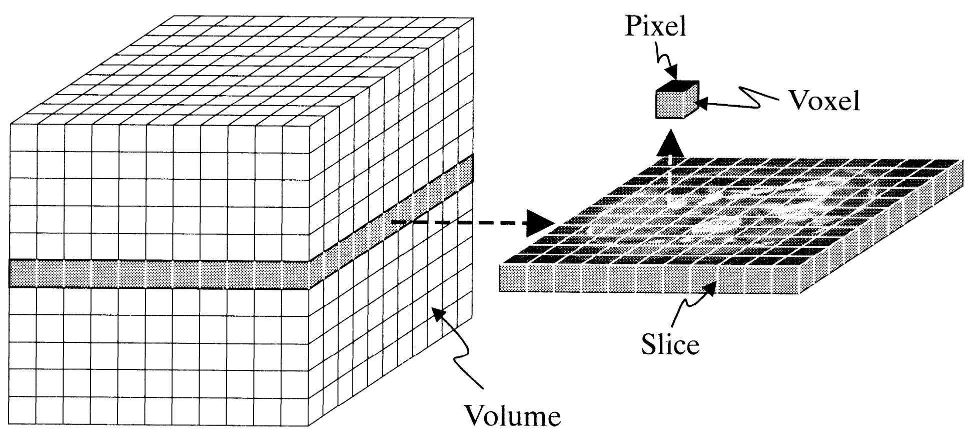
A **continuous** 2-D signal

$$f(x, y), \quad -\infty \le x, y \le \infty$$





- **Discrete signal: Pixel** and **voxel** representations of a continuous signal



?? How do we mathematically model/describe an imaging system?

?? how do we mathematically describe the response of an imaging system to an arbitrary input signal?

Continuous Fourier Transform

- For any square-integrable function *f(x,y)*, a **continuous Fourier transform** is defined as

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$where\ j=\sqrt{-1}$$

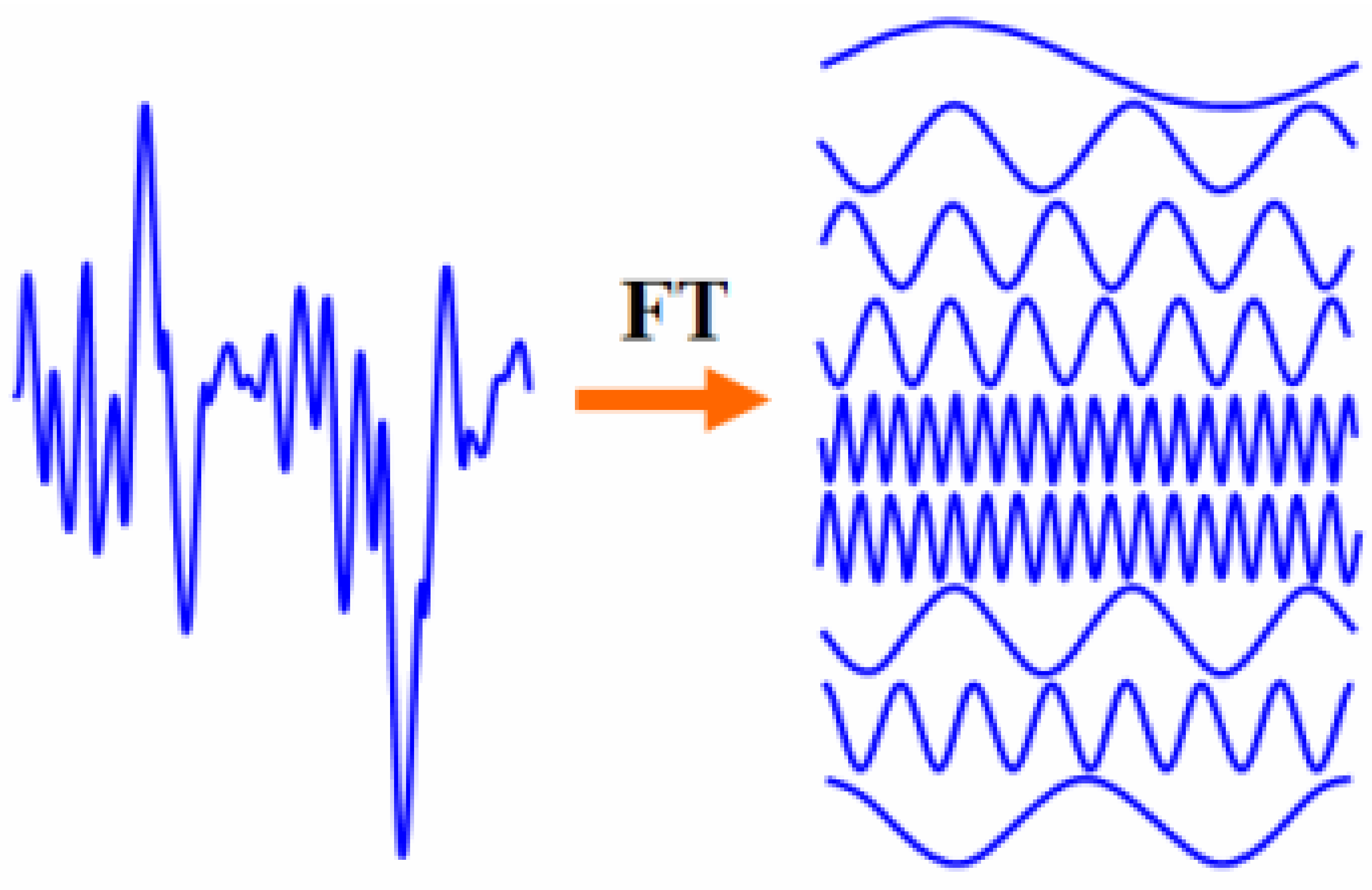
- We can also define an ***inverse Fourier transform*** as

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

- Both f(x,y) and F(u,v) have infinite support.
- Both f(x,y) and F(u,v) are defined on a continuum of values.
- f(x,y) and F(u,v) must contain the same information.

$$e^{-j\cdot 2\pi(ux+vy)} = cos[2\pi(ux+vy)] - j\cdot sin[2\pi(ux+vy)]$$

Continuous Fourier Transform



www.revisemri.com

Discrete Fourier Transform in 1-D

The ***discrete Fourier transform*** (DFT) is defined as

$$F_n = \sum_{k=0}^{N-1} f_k e^{-\frac{j2\pi nk}{N}}, \text{n} = 0,1,2,\ldots\text{N} - 1$$

n = 0 corresponding to the DC component (spatial frenquency is zero)

n = 1,..., N/2 - 1 are corresponding to the positive frenquencies $0 < u < u_c$

n = N/2, ..., N - 1 are corresponding to the negative frenquencies $-u_c < u < 0$

The ***inverse DFT*** is defined as

$$f_k = \frac{1}{N}\sum_{n=0}^{N-1} F_n e^{-\frac{j2\pi nk}{N}}, \text{k} = 0,1,2,\ldots\text{N} - 1$$

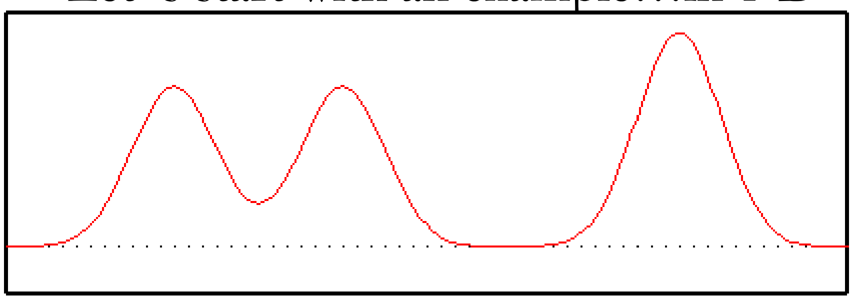
$$e^{-j\frac{2\pi nk}{N}} = cos\left[\frac{2\pi nk}{N}\right] - j \cdot sin[\frac{2\pi nk}{N}]$$

Continuous Fourier Transform

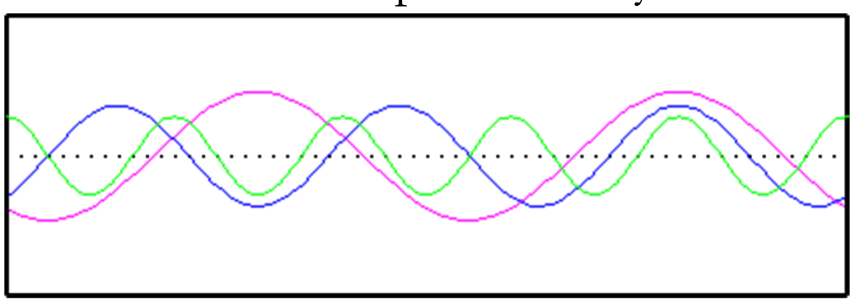
A Fourier Transform is an integral transform that re-expresses a function in terms of different sine waves of varying amplitudes, wavelengths, and phases.

So what does this mean exactly?

Let's start with an example…in 1-D

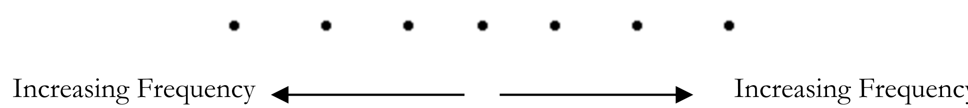


Can be represented by:



When you let these three waves interfere with each other you get your original wave function!

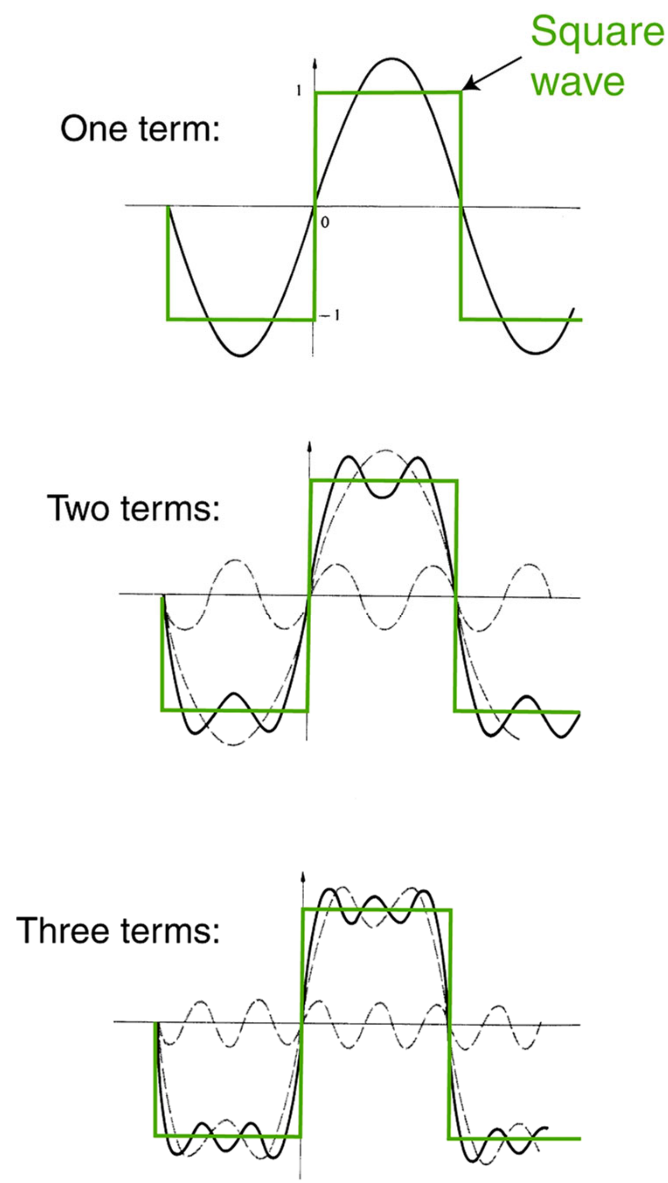
Since this object can be made up of 3 fundamental frequencies an ideal Fourier Transform would look something like this:



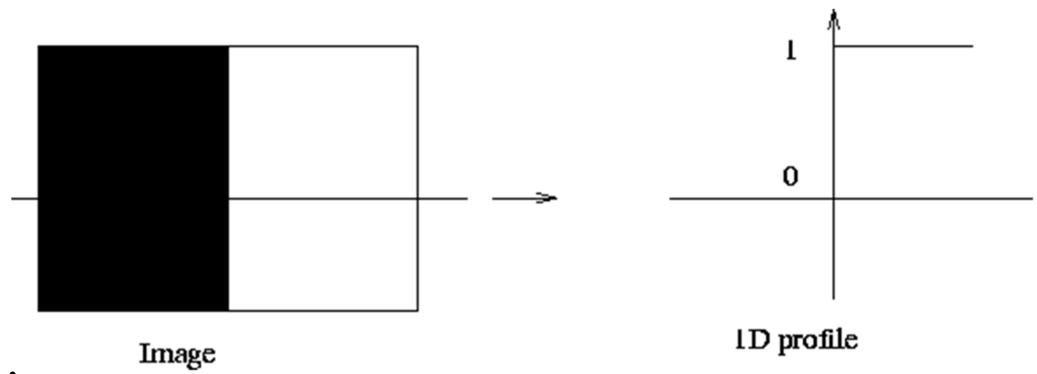
Notice that it is symmetric around the central point and that the amount of points radiating outward correspond to the distinct frequencies used in creating the image.

Fourier Transform and Spatial Frequency

Fourier transform provides information on the sinusoidal composition of a signal at different ***spatial frequencies***.

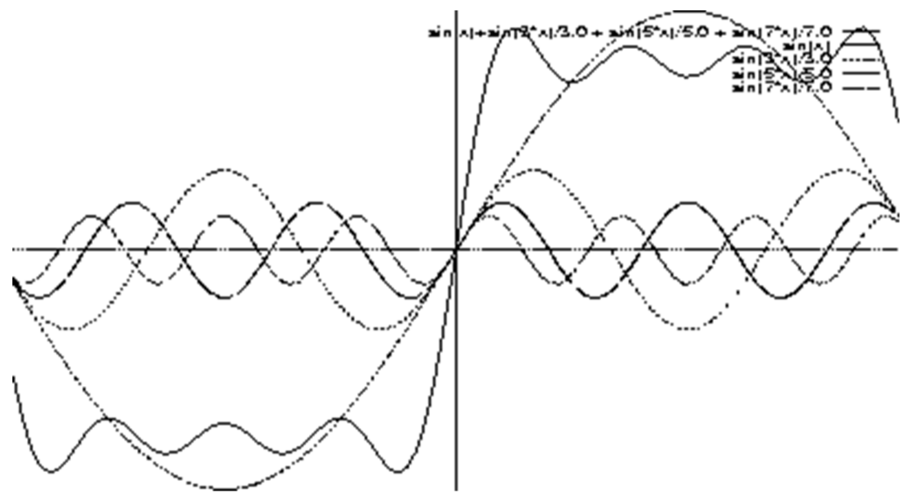


What is Spatial Frequency?



$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)e^{-j2\pi(ux+vy)}dudv$$

$$e^{-j2\pi(ux+vy)}$$
$$=\cos[2\pi(ux+vy)]+j\sin[2\pi(ux+vy)]$$



?? How do we mathematically model/describe an imaging system?

?? how do we mathematically describe the response of an imaging system to an arbitrary input signal?

?? What is a signal anyway, and what is an imaging system anyway?

The Basic Idea for Modeling an Imaging System

The task of analyzing the response of a given system to an arbitrary input signal could be simplified by

- first, **decomposing** the input signal into the linear combination of a series basis functions …
- then figure out the response of the system to the basis signal …
- if we consider the imaging system is a linear system,
- The overall response of the system to the input signal could be synthesized based on the responses of the system to the basis input signals …

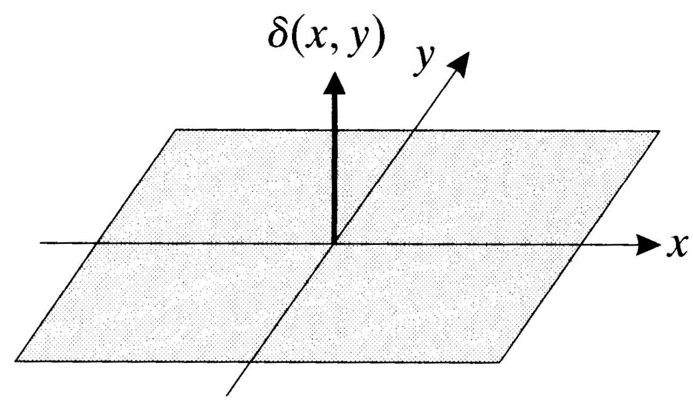
Point Impulse Signal

- A point source is mathematically represented by the **delta function** or **Dirac function**.

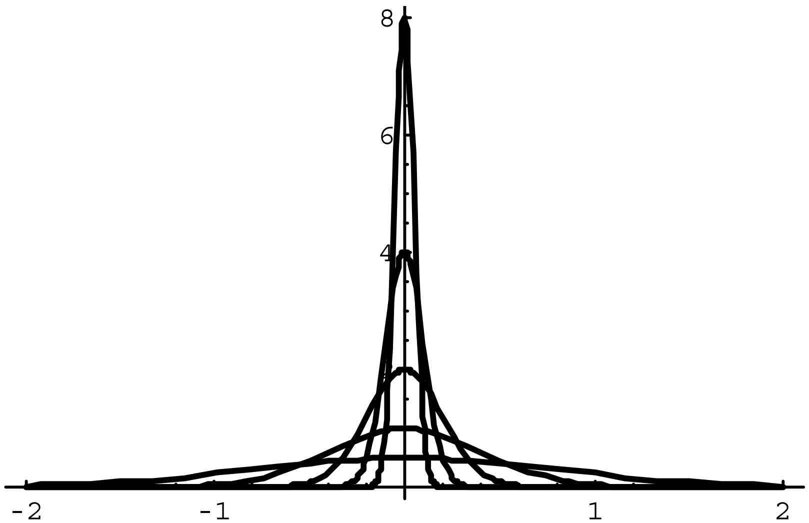
$$\delta(x,y)\begin{cases}\neq 0, & x=0 \; and \; y=0 \\ =0, & otherwise\end{cases}$$

and

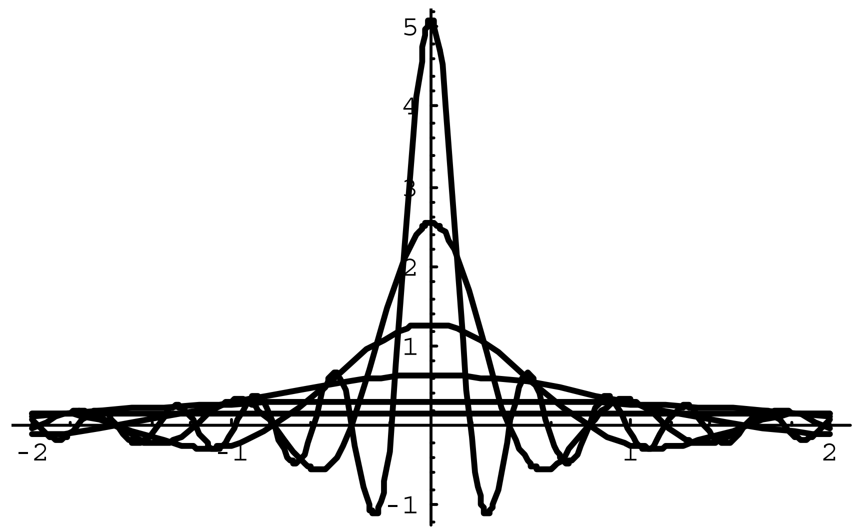
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)dxdy=1$$



Point Impulse Signal



$$\delta(x) = \lim_{\alpha\to\infty} \alpha\ e^{-\pi\alpha^2 x^2}$$



$$\delta(x) = \lim_{\alpha\to\infty} \frac{\alpha}{\pi} \operatorname{sinc}(\alpha x);\ \operatorname{sinc}(x) = \frac{\sin(x)}{x}$$

Point Impulse Signal

- The **sampling property**

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x-\xi, y-\eta)dxdy = f(\xi,\eta)$$

- The **scaling property**

$$\delta(ax,by) = \frac{1}{|ab|}\delta(x,y)$$

$$\delta(x,y)\begin{cases} \neq 0, & x = 0 \; and \; y = 0 \\ = 0, & otherwise \end{cases}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = 1$$

Comb and Sampling Function

- The **2-D comb function**

$$comb(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m, y-n)$$

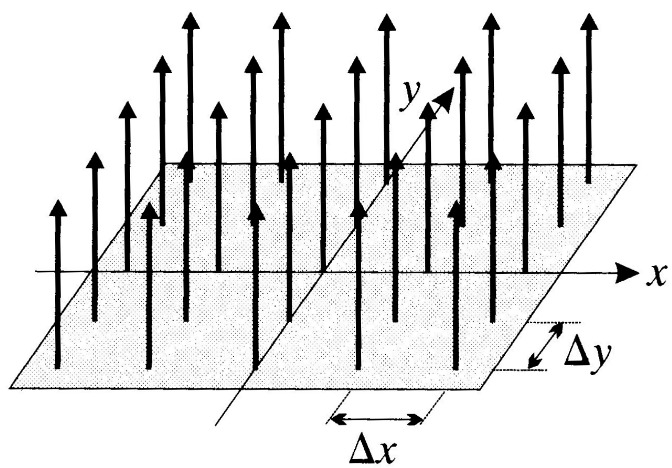
Comb and Sampling Function

- The **2-D sampling function**

$$\delta_s(x,y,\Delta x,\Delta y)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m\Delta x,y-n\Delta y)$$

where Δx and Δy are the sampling intervals

$$\delta_s(x,y,\Delta x,\Delta y)=\frac{1}{\Delta x\Delta y}comb\left(\frac{x}{\Delta x},\frac{y}{\Delta y}\right)$$



$$comb(x,y)=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m,y-n)$$

$$\delta(ax,by)=\frac{1}{|ab|}\delta(x,y)$$

Comb and Sampling Function

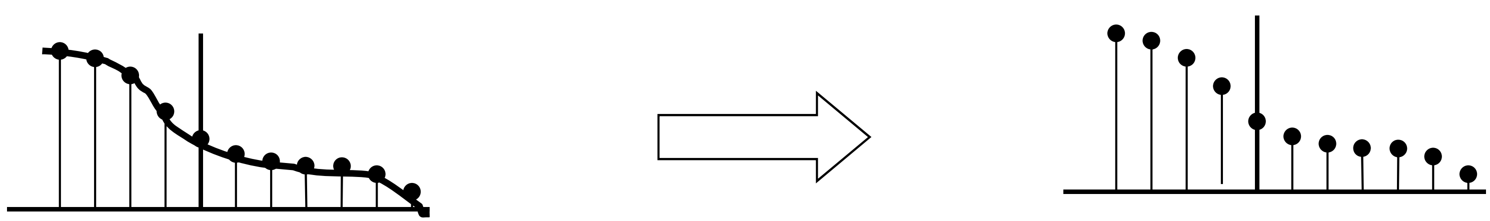
- The sampling function is critical for the **discretization** of continuous signals.
- The sampled signal function is then

$$\delta_s(x, y, \Delta x, \Delta y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$$

where Δx and Δy are the sampling intervals

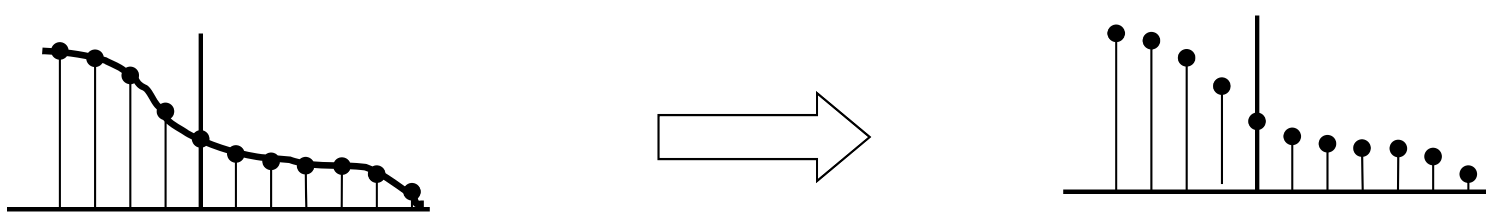
$$f_s(x, y) = f(x, y) \cdot \delta_s(x, y)$$

$$= f(x, y) \cdot \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$$



A central question about sampling:

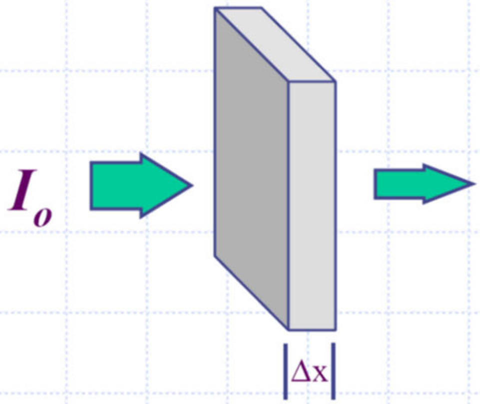
Will this continuous-to-discrete sampling process cause any loss in information?



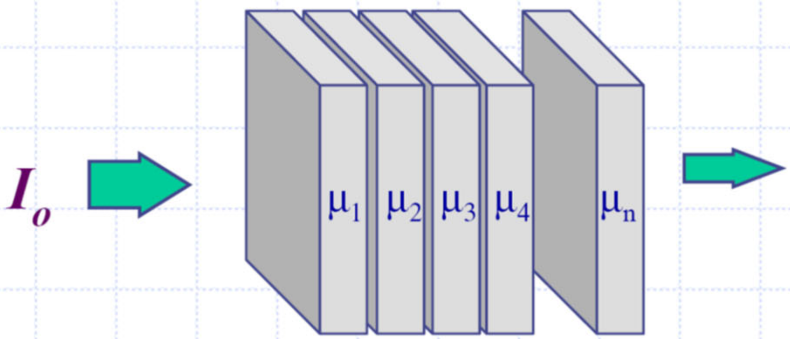
Revisit to X-ray Planar Radiography

What are we measuring with planar X-ray radiography?

2 % attenuation change detectable in film



$$I = I_o e^{-\mu \Delta x}$$

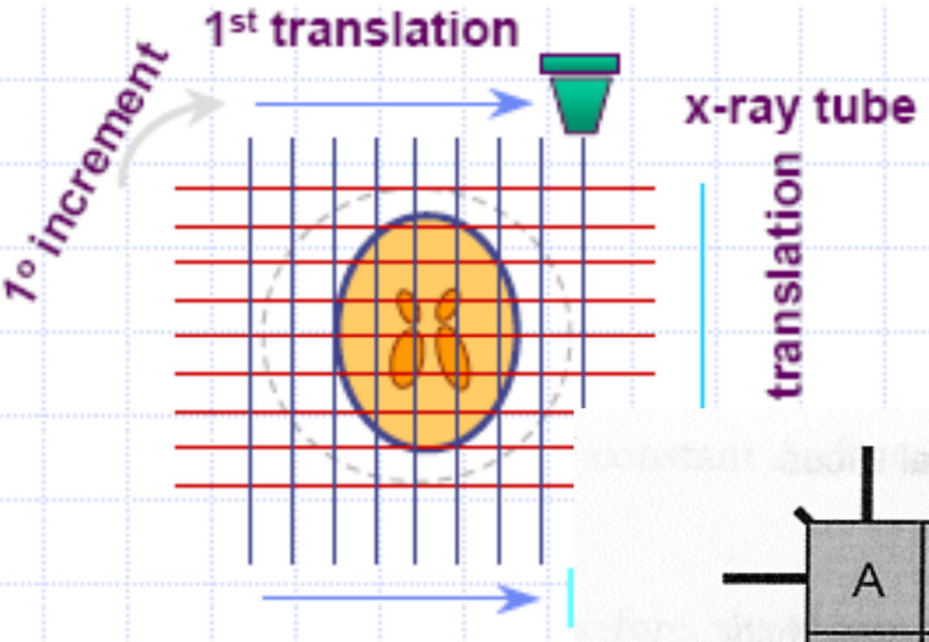


$$I = I_o e^{-\mu_1 \Delta x} e^{-\mu_2 \Delta x} \cdots e^{-\mu}$$

$$= I_o e^{-(\mu_1 + \mu_2 + \cdots + \mu_n)\Delta x}$$

$\Delta x \rightarrow 0,$ $\quad P = -\ln\left(\frac{I}{I_o}\right) = \int_{-\infty}^{\infty} \mu(x)\, dx$

X-ray Computed Tomography (CT)



0.2 % attenuation change detectable in CT Images !!

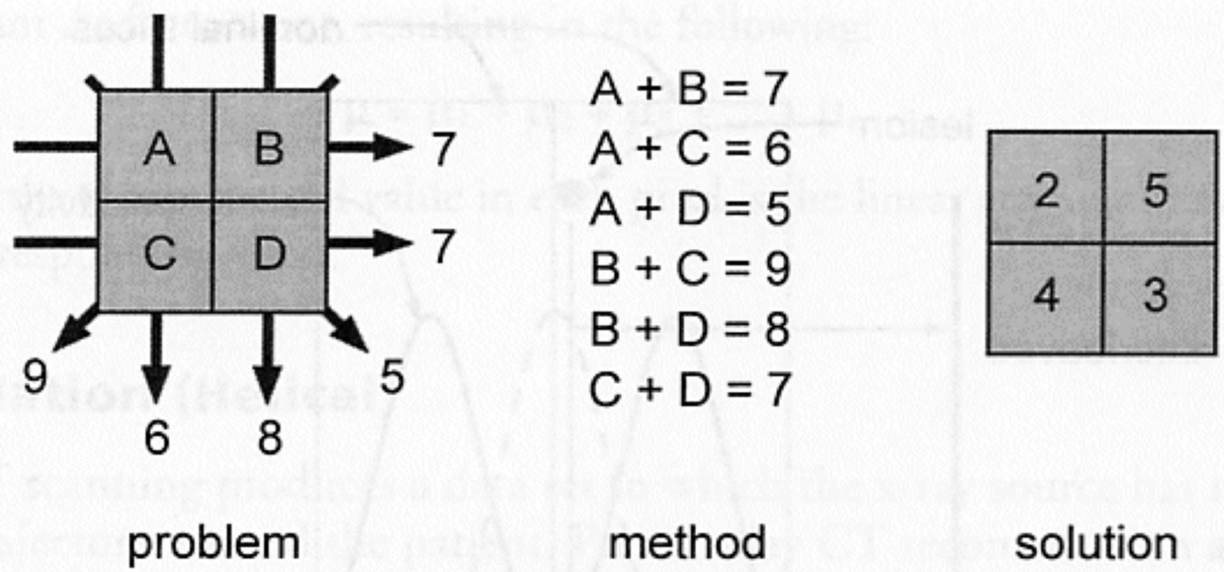
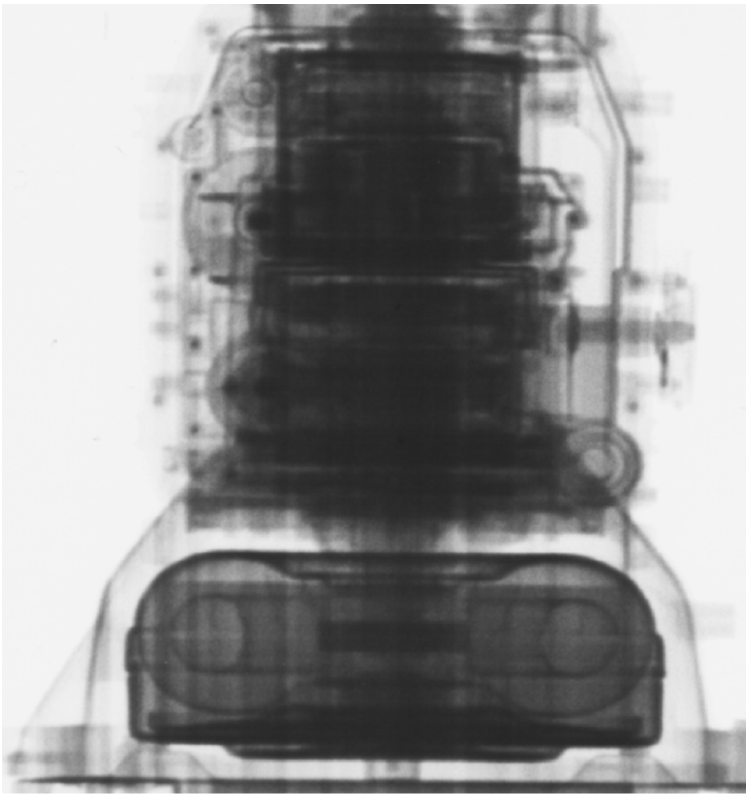


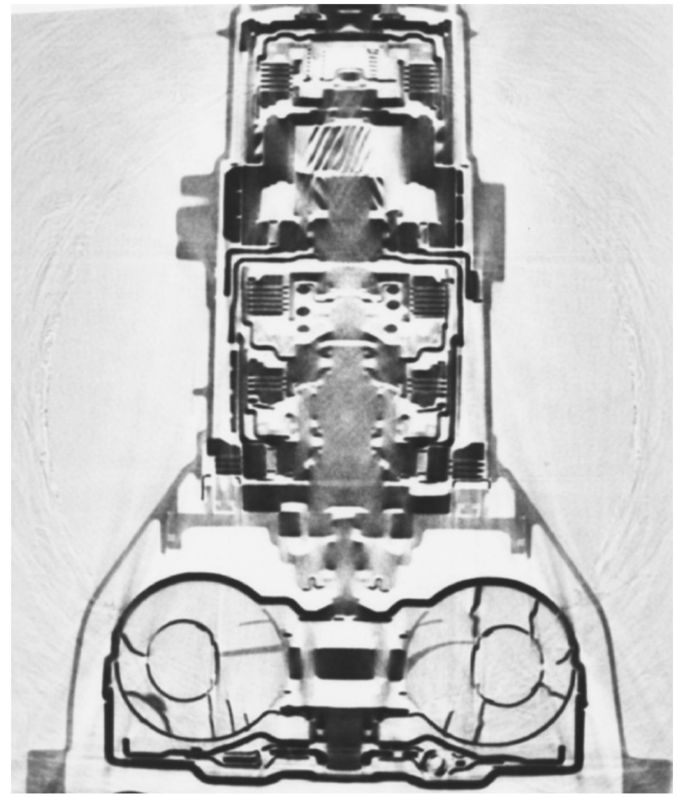
FIGURE 13-27. The mathematical problem posed by computed tomographic (CT) reconstruction is to calculate image data (the pixel values—A, B, C, and D) from the projection values *(arrows)*. For the simple image of four pixels shown here, algebra can be used to solve for the pixel values. With the six equations shown, using substitution of equations, the solution can be determined as illustrated. For the larger images of clinical CT, algebraic solutions become unfeasible, and filtered backprojection methods are used.

X-ray Computed Tomography (CT)

Planar X-Ray



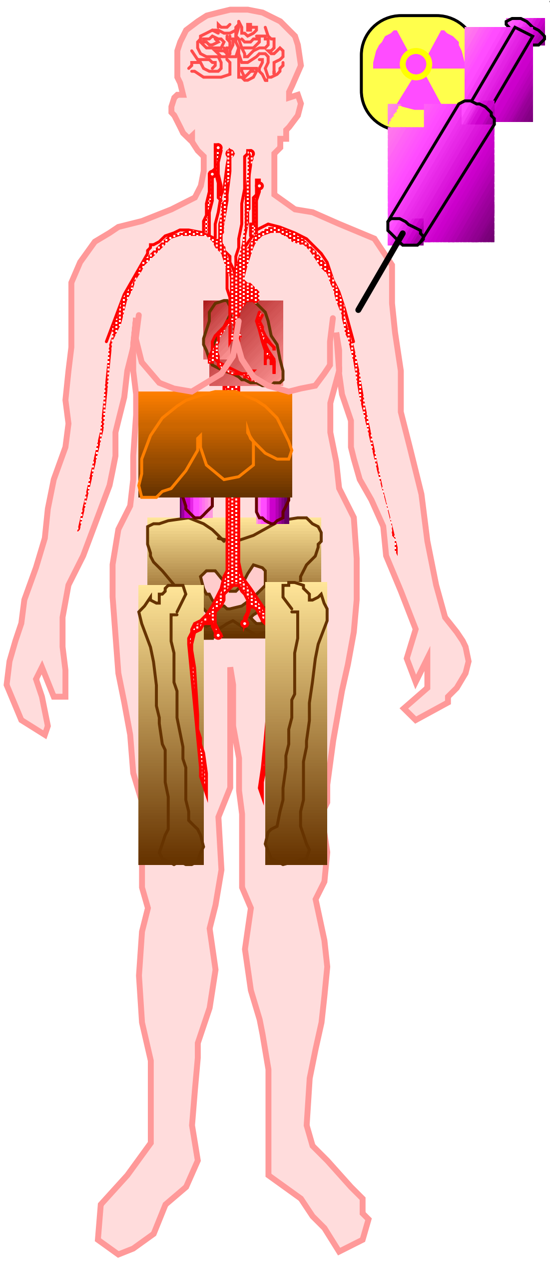
Computed Tomography



Separates Objects on Different Planes

Images courtesy of Robert McGee, Ford Motor Company

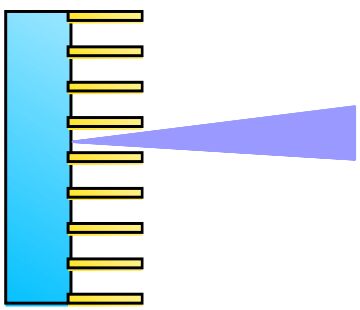
Emission Tomography



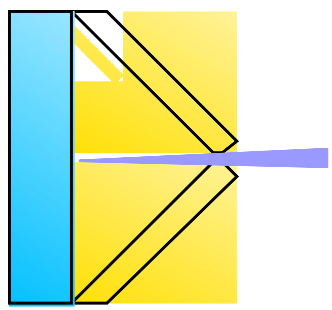
- Drug is labeled with radioisotopes that emit gamma rays.
- Drug localizes in patient according to metabolic properties of that drug.
- Trace (pico-molar) quantities of drug are sufficient.
- Radiation dose fairly small (<1 rem).

Drug Distributes in Body

Single Photon Emission Computed Tomography (SPECT)



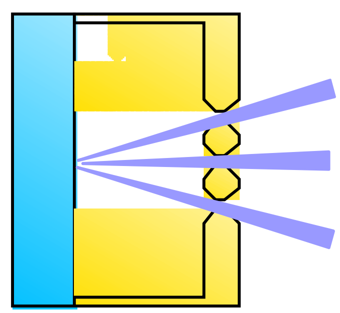
Collimator



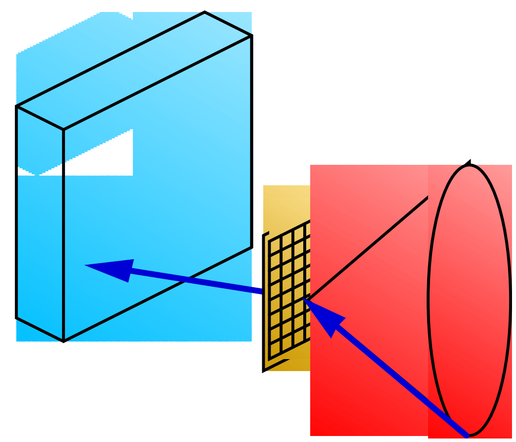
Pinhole

Collimator in front of the detector to select gamma rays from certain directions only …

Rotated around the object for collecting multiple projections …

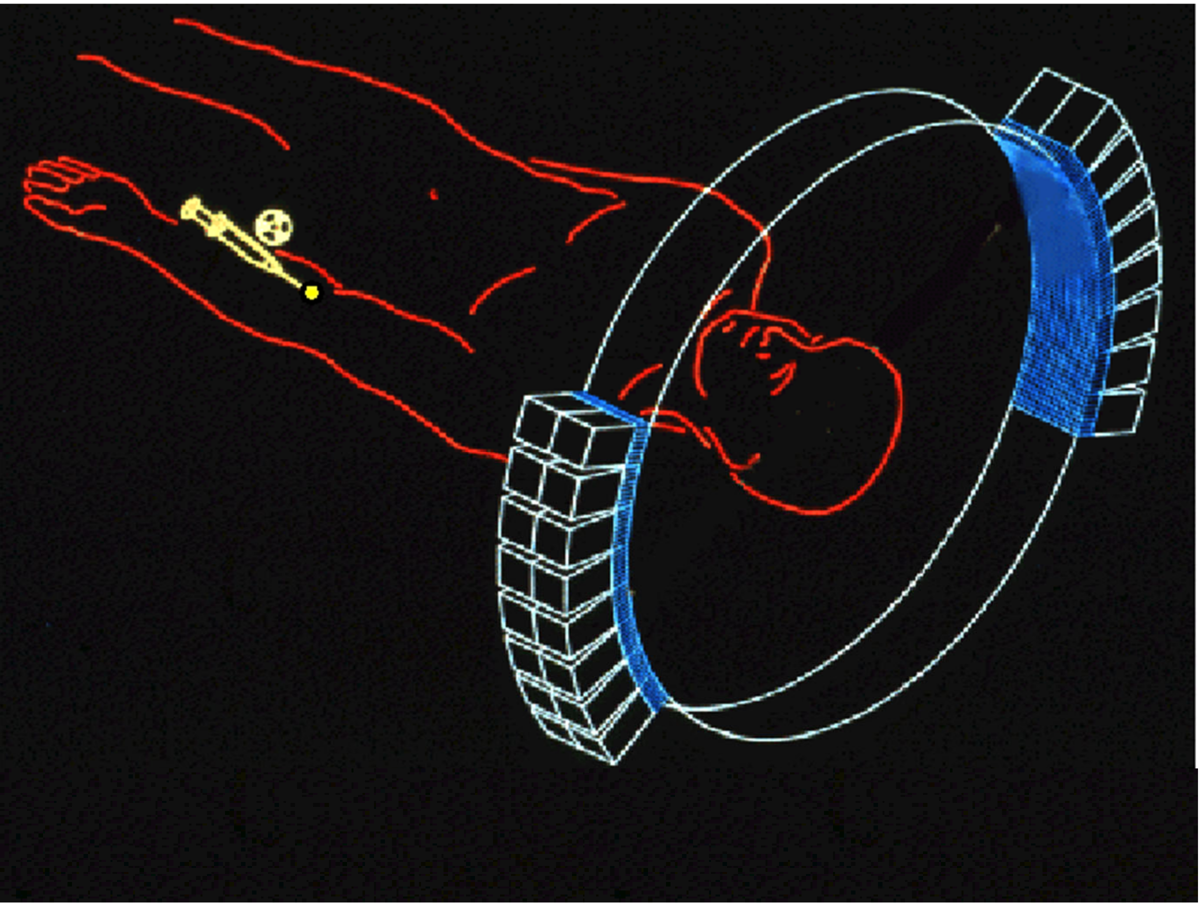


Coded Aperture



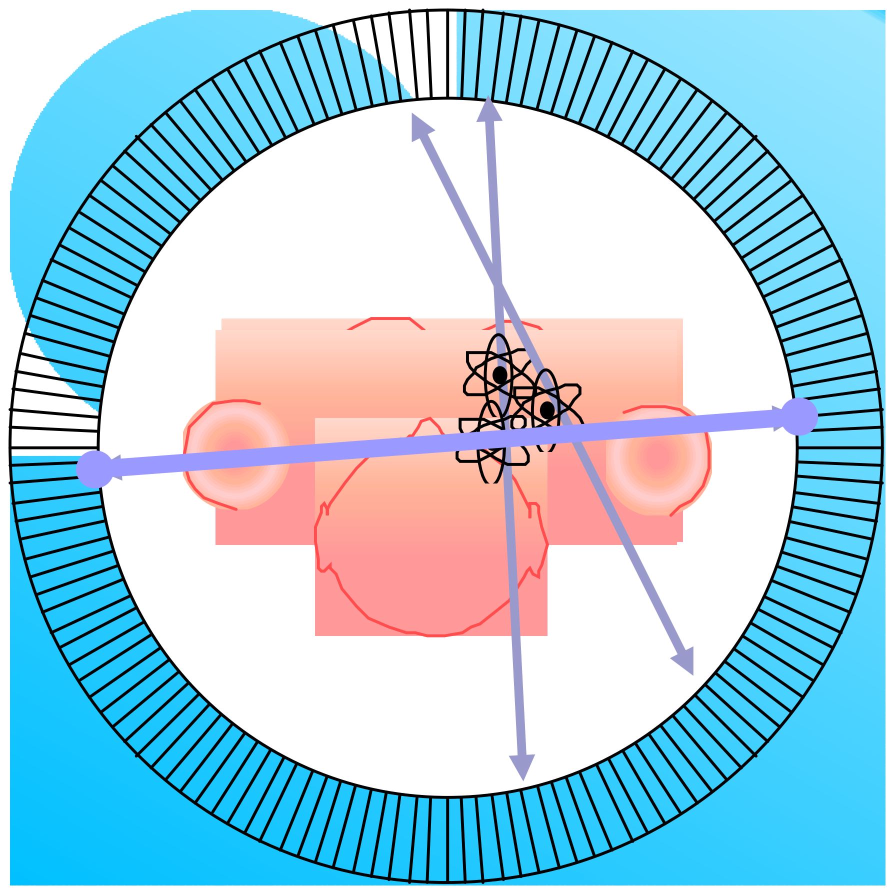
Compton

Positron Emission Tomography



Typical Detection Process

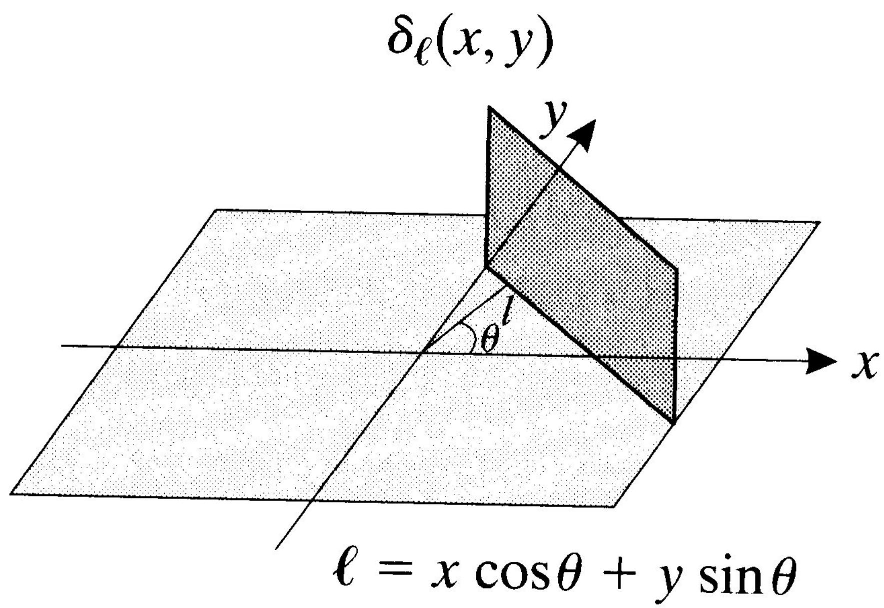
Collection of Line-integrals



Line Impulse Signal (1)

$$\delta_L(x, y) = \delta(x\cos\theta + y\sin\theta - l)$$

$$where\ \delta(x) = \begin{cases} >0, & x\cos\theta + y\sin\theta = l \\ 0, & otherwise \end{cases}$$



Line Impulse Signal (1)

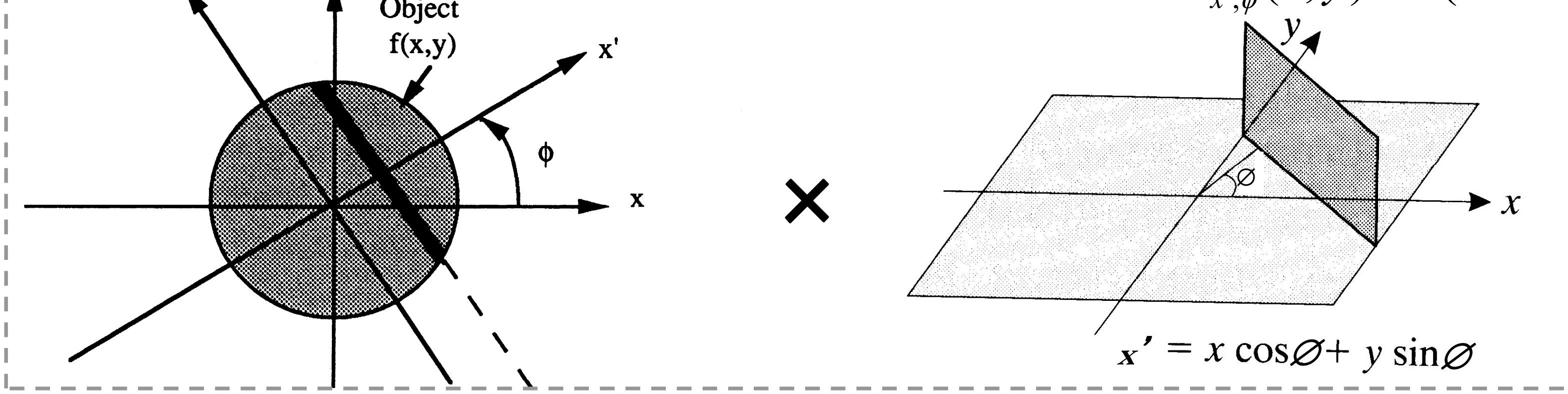
$$\delta_L(x, y) = \delta(x\cos\theta + y\sin\theta - l)$$

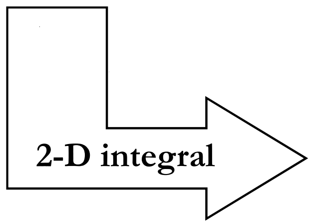
- It can be used to measure the spatial resolution of a given imaging system.

- It is used to calculate the line-integral projection data for a given 2-D object.

Line Impulse Signal (2)

$$\delta_{x',\phi}(x,y)=\delta(x\cos\phi+y\sin\phi-x')$$





The value of the projection function $p_\phi(x')$ at this point is the integral of the function of f(x,y) along the straight line: $x'=x\cos\phi+y\sin\phi$

The integral of the product of a line impulse function and a given 2-D signal gives the ***projection*** data from a given view …

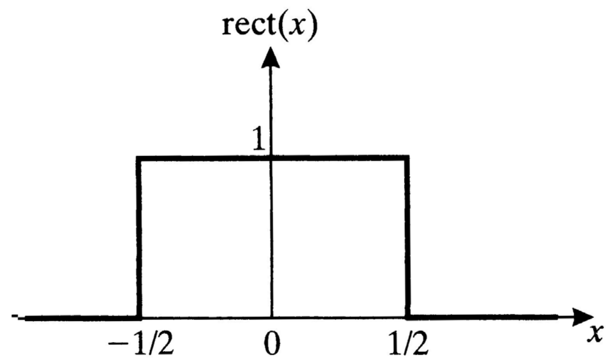
$$p_\phi(x')=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\phi+y\sin\phi-x')dxdy$$

Line-impulse function is the key for modeling the projection process that underlying tomographic imaging process …

Rect Function

- **Rect** function:

$$rect(x,y)=\begin{cases}1, \text{ for } |x|<\frac{1}{2} \text{ and } |y|<\frac{1}{2}\\ 0, \text{ otherwise}\end{cases}$$



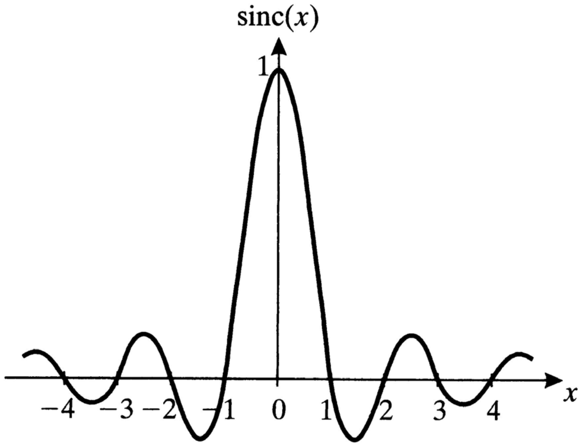
- It is normally used to pick up a particulate section of a given function:

$$f(x,y)\cdot rect\left(\frac{x-\xi}{w_X},\frac{y-\eta}{w_Y}\right)$$

Sinc Function

- The **sinc** function is defined as

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



- The **sinc** function is **normalized**.

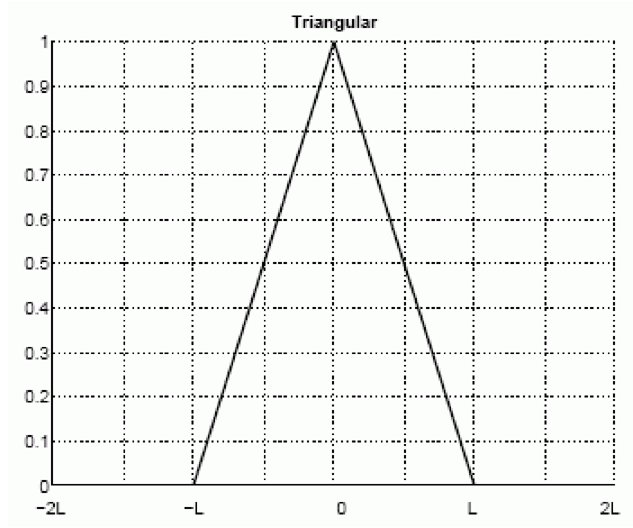
$$\int_{-\infty}^{\infty} \mathrm{sinc}(x)dx = 1$$

Any arbitrary band-limited signal can be written as a weighted sum of multiple sinc functions … (the **Nyquist Sampling Theorem**)

Triangular Signals and Gaussian Signals

- Triangular function:

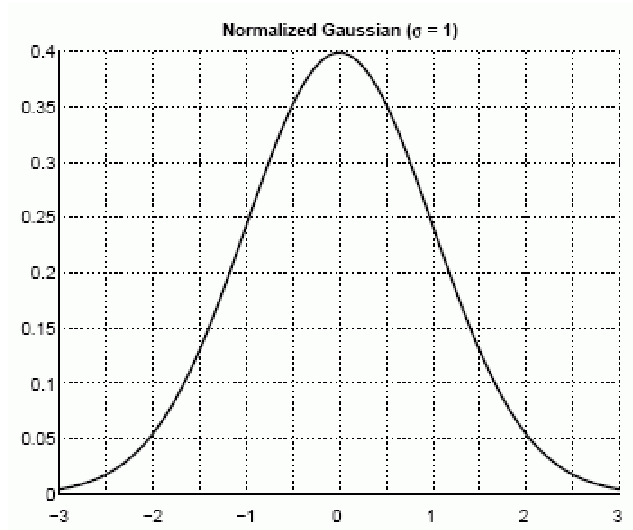
$$Tri\left(\frac{x}{2L}\right) = 1 - \frac{|x|}{L} \quad \text{for } |x| < L$$
$$= 0 \quad \text{for } |x| > L$$



- Normalized Gaussian function:

$$G_{1D}(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma^2}}$$

$$G_{2D}(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$



Separable Signals and Periodic Signals

- The **separable signals** is a class of continuous signals that satisfy

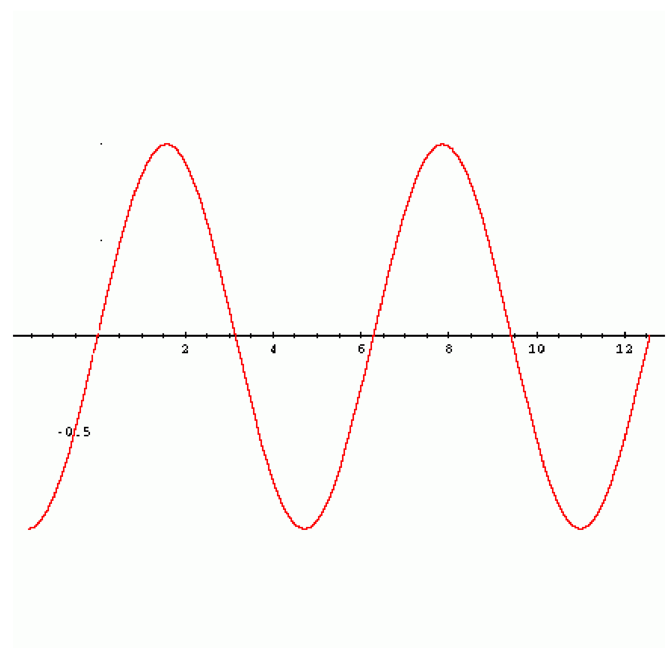
$$\frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}}$$

$$f(x,y) = f_1(x) \cdot f_2(y)$$

- A signal is **periodic** if

$$f(x,y) = f(x+X,y) = f(x,y+Y)$$

where X and Y are the signal periods

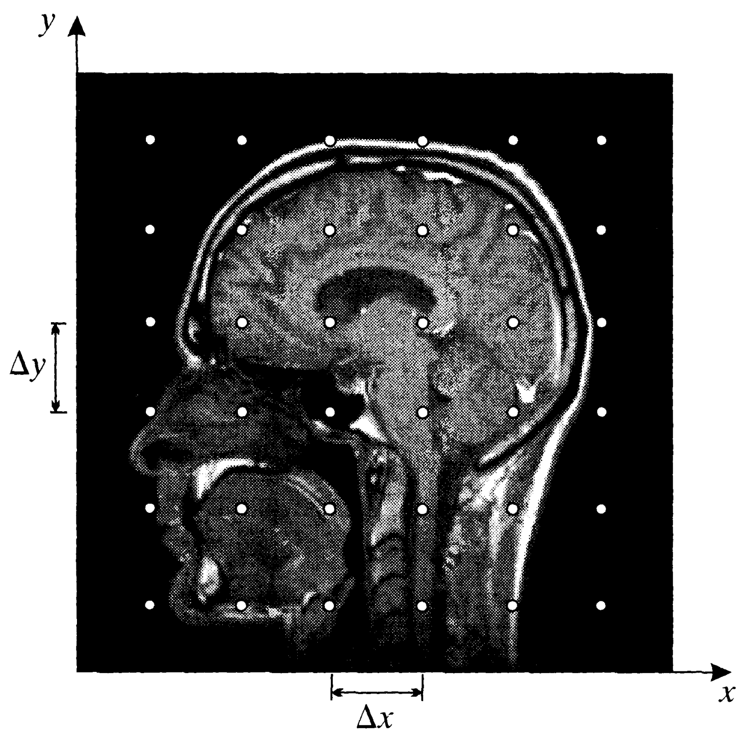


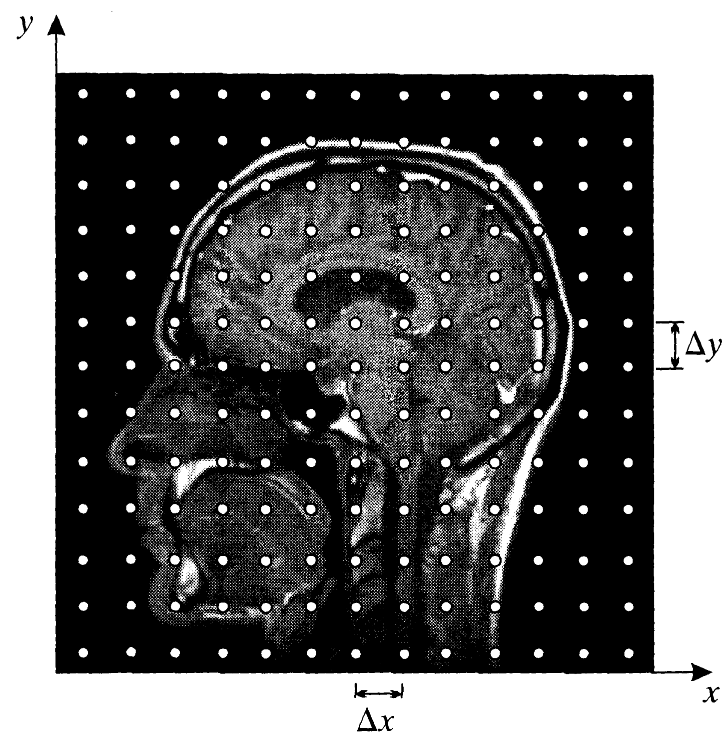
Two Dimensional Sampling

$$f_s(x,y) = f(x,y)\cdot\delta_s(x,y,\Delta x,\Delta y)$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(x,y)\cdot\delta(x-n\Delta x, y-m\Delta y)$$

$$= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(n\Delta x, m\Delta y)\cdot\delta(x-n\Delta x, y-m\Delta y)$$



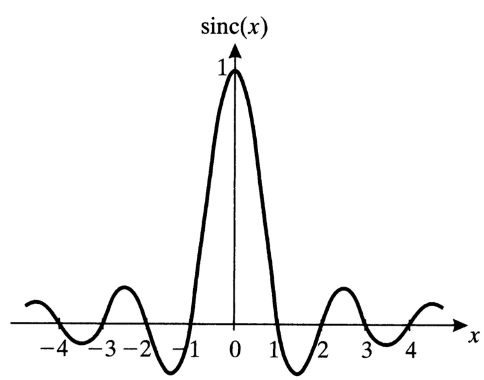


Restoration of the Original 2-D Function

Given that the Nyquist sampling condition is met, the original function could be recovered exactly as

$$
\begin{aligned}
f(x,y) &= f_s(x,y) * h(x,y) \\
&= f_s(x,y) * \left[\frac{1}{\Delta x}\cdot \text{sinc}\left(\frac{x}{\Delta x}\right)\right]\left[\frac{1}{\Delta y}\cdot \text{sinc}\left(\frac{y}{\Delta y}\right)\right] \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} f(n\Delta x, m\Delta y)\cdot \delta(x-n\cdot\Delta x, y-m\cdot\Delta y) * \left\{\left[\frac{1}{\Delta x}\cdot \text{sinc}\left(\frac{x}{\Delta x}\right)\right]\left[\frac{1}{\Delta y}\cdot \text{sinc}\left(\frac{y}{\Delta y}\right)\right]\right\} \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{\Delta x\Delta y} f(n\Delta x, m\Delta y)\cdot \text{sinc}\left(\frac{x-n\cdot\Delta x}{\Delta x}\right)\cdot \text{sinc}\left(\frac{y-m\cdot\Delta y}{\Delta y}\right)
\end{aligned}
$$

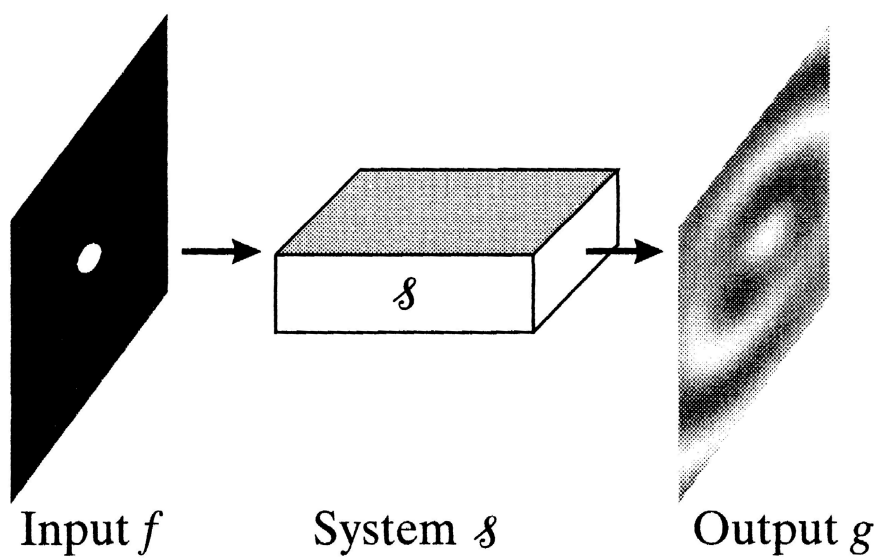
$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



General Concept of a System

- A **continuous-to-continuous system** is defined as

$$g(x, y) = \mathcal{S}[f(x, y)]$$



- A system is a **mapping process** from an input signal to the output signal

Linear Systems

A system is **linear** if it satisfies the **superposition principle**

$$S\left[f(x,y)=\sum_{i=1}^{I} w_i \cdot f_i(x,y)\right]=\sum_{i=1}^{I} w_i \cdot S\left[f_i(x,y)\right]$$

where

$f(x,y)$ is the input signal,

$S[\cdot]$ is an operator that represents the system,

$f(x,y)$ is the total input signal and

w_is are weighting factors.

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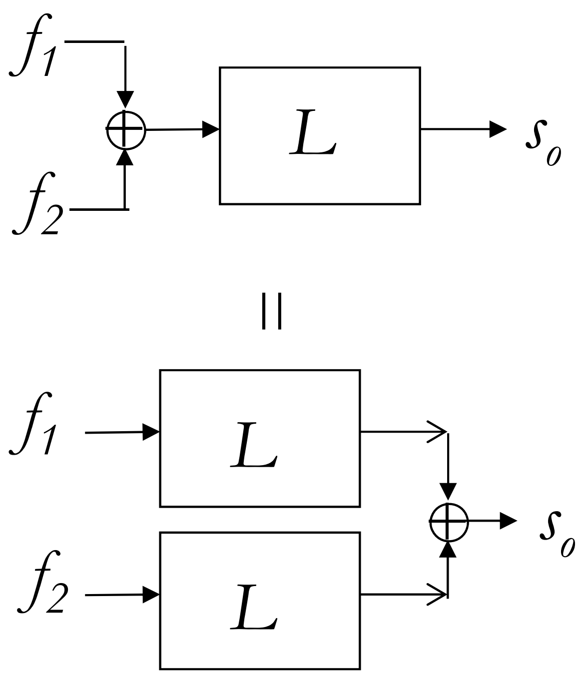
Linear Systems – An Example

For example, consider an ***amplifier*** with gain A:



$$S[w_1 f_1 + w_2 f_2] = A(w_1 f_1 + w_2 f_2)$$
$$= A w_1 f_1 + A w_2 f_2 = w_1 S[f_1] + w_2 S[f_2]$$

It satisfies the Superposition Principle

$$S\left[f(x,y) = \sum_{i=1}^{I} w_i \cdot f_i(x,y) \right] = \sum_{i=1}^{I} w_i \cdot S[f_i(x,y)]$$

Covered in lecture

Linear Systems – Why Important?

- Linear systems is mathematically more "***tractable***".

$$f(x,y) \Leftarrow S\left[f(x,y) = \sum_{i=1}^{I} w_i \cdot f_i(x,y) \right] \Rightarrow g(x,y)$$

- Many imaging systems used in medical and other applications can be described as linear systems.

Covered in lecture

Linear Systems – Why Important?

- Linear systems satisfy the Superposition Principle.

$$g(x,y) = S[f(x,y)] = S\left[\sum_{i=1}^{I} w_i \cdot f_i(x,y)\right] = \sum_{i=1}^{I} w_i \cdot S[f_i(x,y)]$$

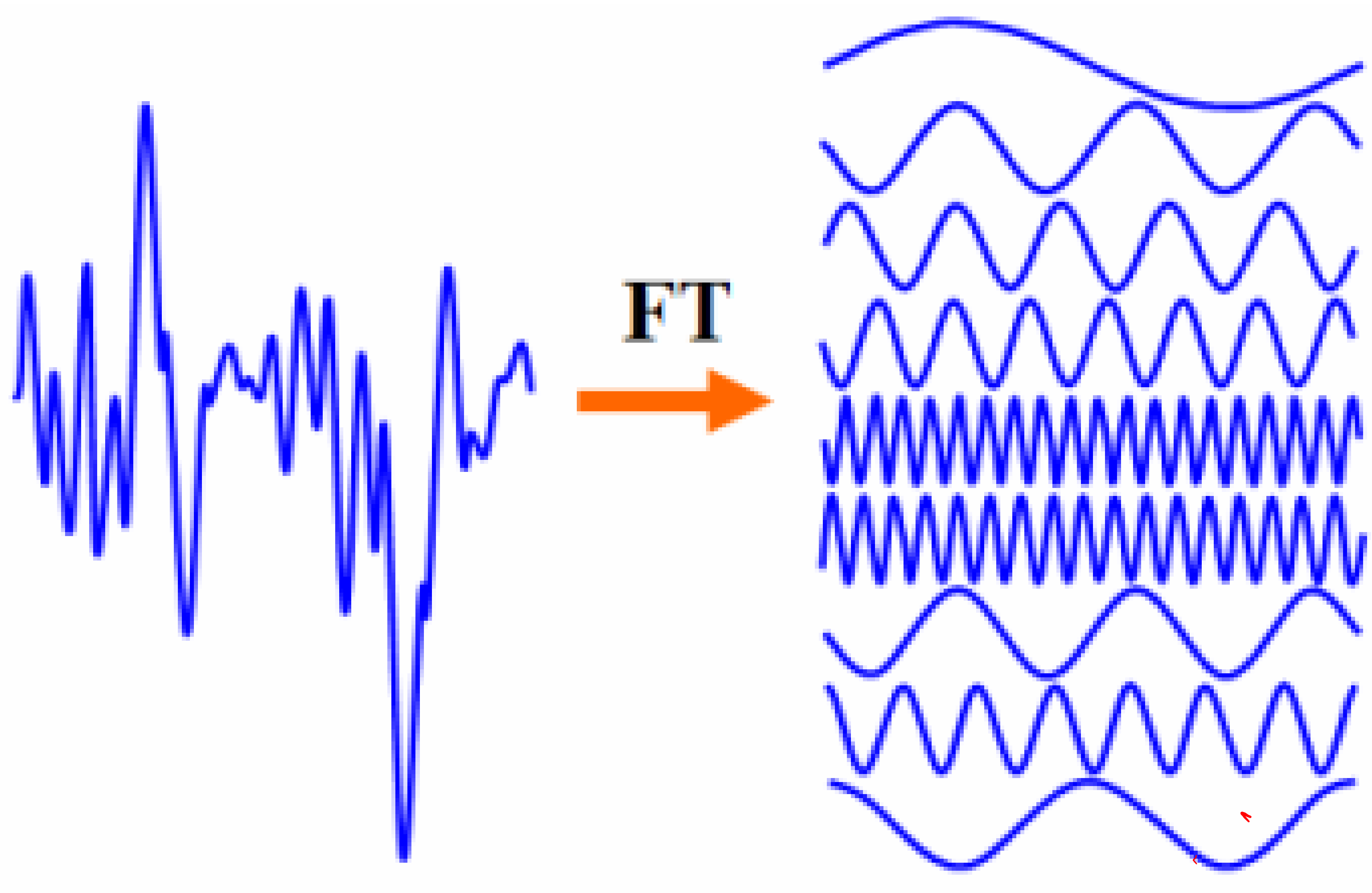
- It would be good if we can decompose an arbitrary signal into a linear combination of a series of basis functions – such as the δ-function.

- If one can derive the response of the system to this basis function,

- then the response of a system to the arbitrary input signal should easily follow …

Covered in lecture

Continuous Fourier Transform



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Comb and Sampling Function

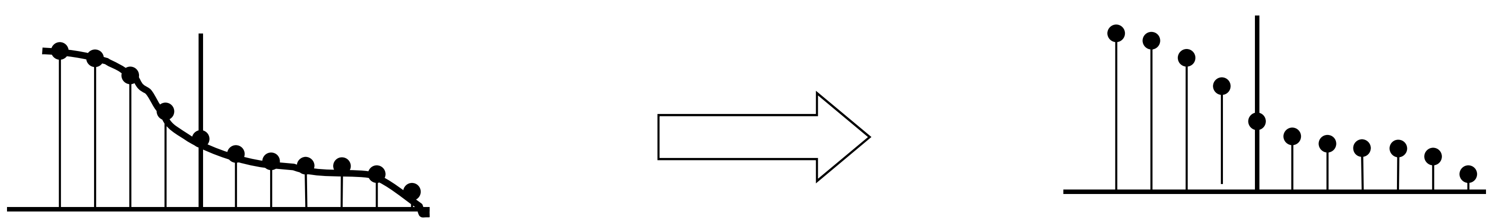
- The sampling function is critical for the **discretization** of continuous signals.
- The sampled signal function is then

$$\delta_s(x, y, \Delta x, \Delta y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$$

where Δx and Δy are the sampling intervals

$$f_s(x, y) = f(x, y) \cdot \delta_s(x, y)$$

$$= f(x, y) \cdot \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$$



Covered in lecture

Linear Systems – Why Important?

Since we can often decompose an arbitrary input signal as a linear combination of basis functions (delta functions, or sinusoidal functions, or sinc functions etc.),

the response of a linear system to the given arbitrary input signal can therefore be modeled as the linear combination of the response of the system to each individual basis functions….

Covered in lecture

Linear Systems – Why Important?

- The Superposition Principle.

$$g(x,y) = S[f(x,y)] = S\left[\sum_{i=1}^{I} w_i \cdot f_i(x,y)\right] = \sum_{i=1}^{I} w_i \cdot S[f_i(x,y)]$$

- Given a discrete input signal

$$f_s(x,y) = f(x,y) \cdot s(x,y)$$

$$= \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} [f(m\Delta x, n\Delta y) \cdot \delta(x - m\Delta x, y - n\Delta y)]$$

- The response of the linear system is

$$g(x,y) = S[f_s(x,y)]$$

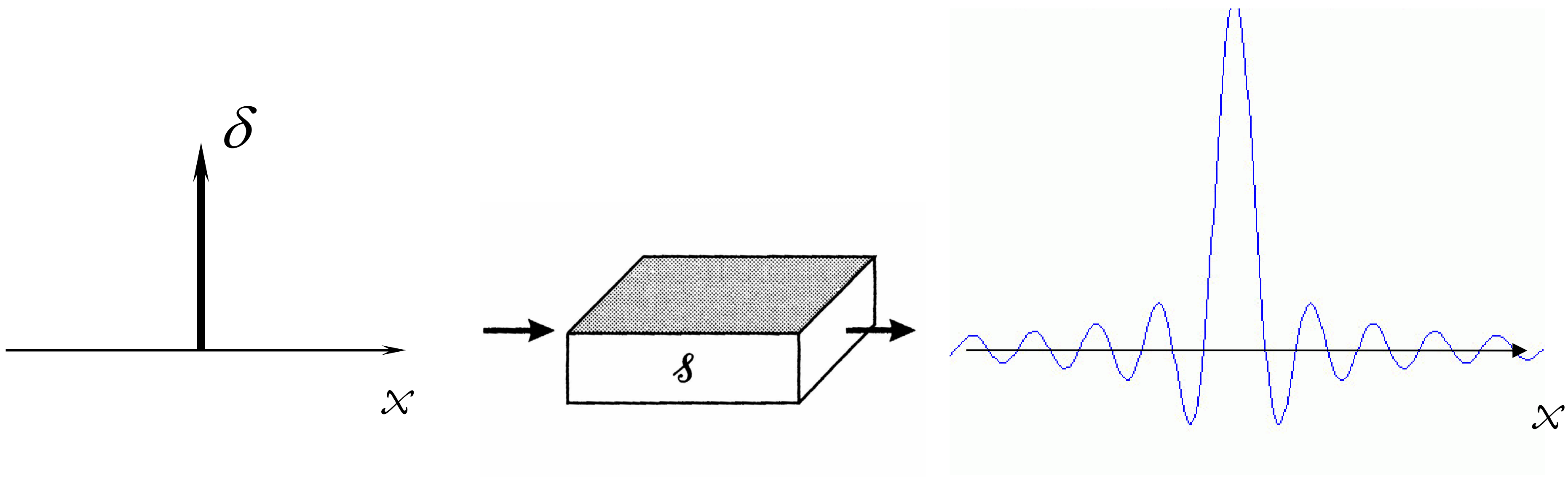
$$= S\left[\frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} [f(m\Delta x, n\Delta y) \cdot \delta(x - m\Delta x, y - n\Delta y)]\right]$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[\frac{1}{\Delta x \Delta y} f(m\Delta x, n\Delta y) \cdot S[\delta(x - m\Delta x, y - n\Delta y)]\right]$$

Covered in lecture

Impulse Response Function

One of the most common shape for impulse responses used in imaging application



Covered in lecture

Impulse Response Function

For a linear system, knowing the IRF, one could compute the output from any arbitrary input function as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x-\xi,y-\eta)dxdy = f(\xi,\eta)$$

or

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\delta(\xi-x,\eta-y)dxdy = f(x,y)$$

$$\Leftrightarrow$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\delta(x-\xi,y-\eta)dxdy = f(x,y)$$

Using the sampling property of delta function

$$g(x,y) = \mathcal{S}[f(x,y)]$$

$$= \mathcal{S}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\delta(x-\xi,y-\eta)d\xi d\eta\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{S}\left[f(\xi,\eta)\delta(x-\xi,y-\eta)\right]d\xi d\eta$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\mathcal{S}\left[\delta(x-\xi,y-\eta)\right]d\xi d\eta$$

The linearity condition is used here

Covered in lecture

Impulse Response Function

For a linear system, knowing the IRF enables one to compute the output from any arbitrary input function.

$$
\begin{aligned}
g(x,y) &= \mathcal{S}[f(x,y)] \\
&= \mathcal{S}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\delta(x-\xi,y-\eta)d\xi d\eta\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{S}\left[f(\xi,\eta)\delta(x-\xi,y-\eta)\right]d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\mathcal{S}\left[\delta(x-\xi,y-\eta)\right]d\xi d\eta
\end{aligned}
$$

The **impulse response function is defined as**

$$h(x,y,\xi,\eta) \equiv \mathcal{S}\left[\delta(x-\xi,y-\eta)\right]$$

Covered in lecture

Impulse Response Function

For a linear system, knowing the IRF enables one to compute the **output from any arbitrary input function**:

$$
\begin{aligned}
g(x,y) &= \mathcal{S}[f(x,y)] \\
&= \mathcal{S}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\delta(x-\xi,y-\eta)d\xi d\eta\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathcal{S}\left[f(\xi,\eta)\delta(x-\xi,y-\eta)\right]d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\mathcal{S}\left[\delta(x-\xi,y-\eta)\right]d\xi d\eta
\end{aligned}
$$

Or written explicitly as

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)h(x,y,\xi,\eta)d\xi d\eta$$

Covered in lecture

Impulse Response Function

- For a 2-D problem, the impulse response is a ***4-D*** function.

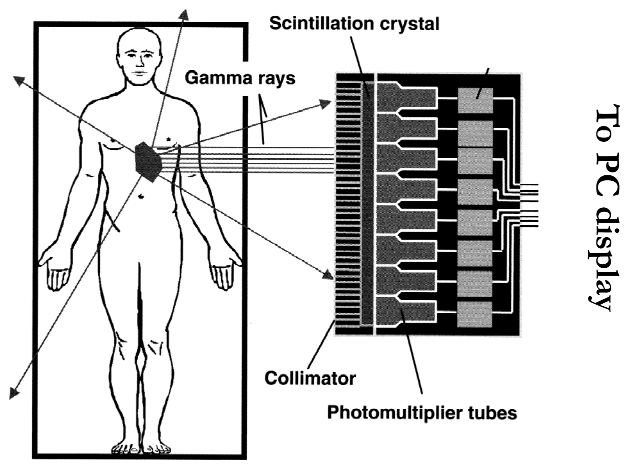
$$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)h(x,y,\xi,\eta)d\xi d\eta$$

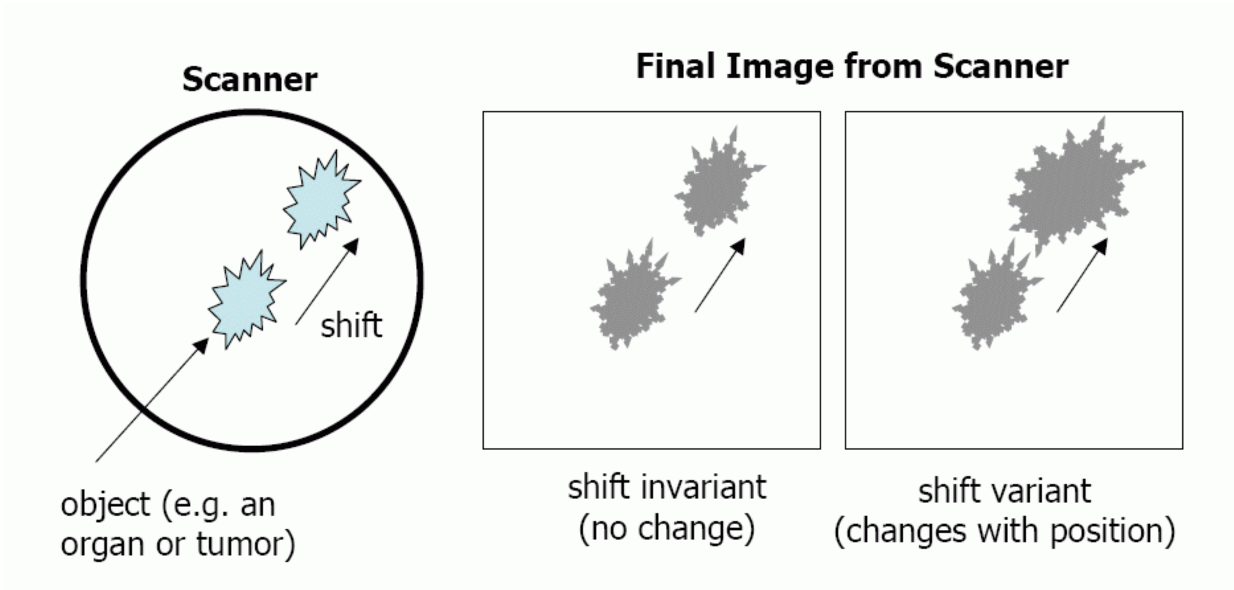
- The computation can be greatly reduced with further simplifications …

What exactly is this function again? What does it tell us about the system?

Covered in lecture

Shift Invariant Systems





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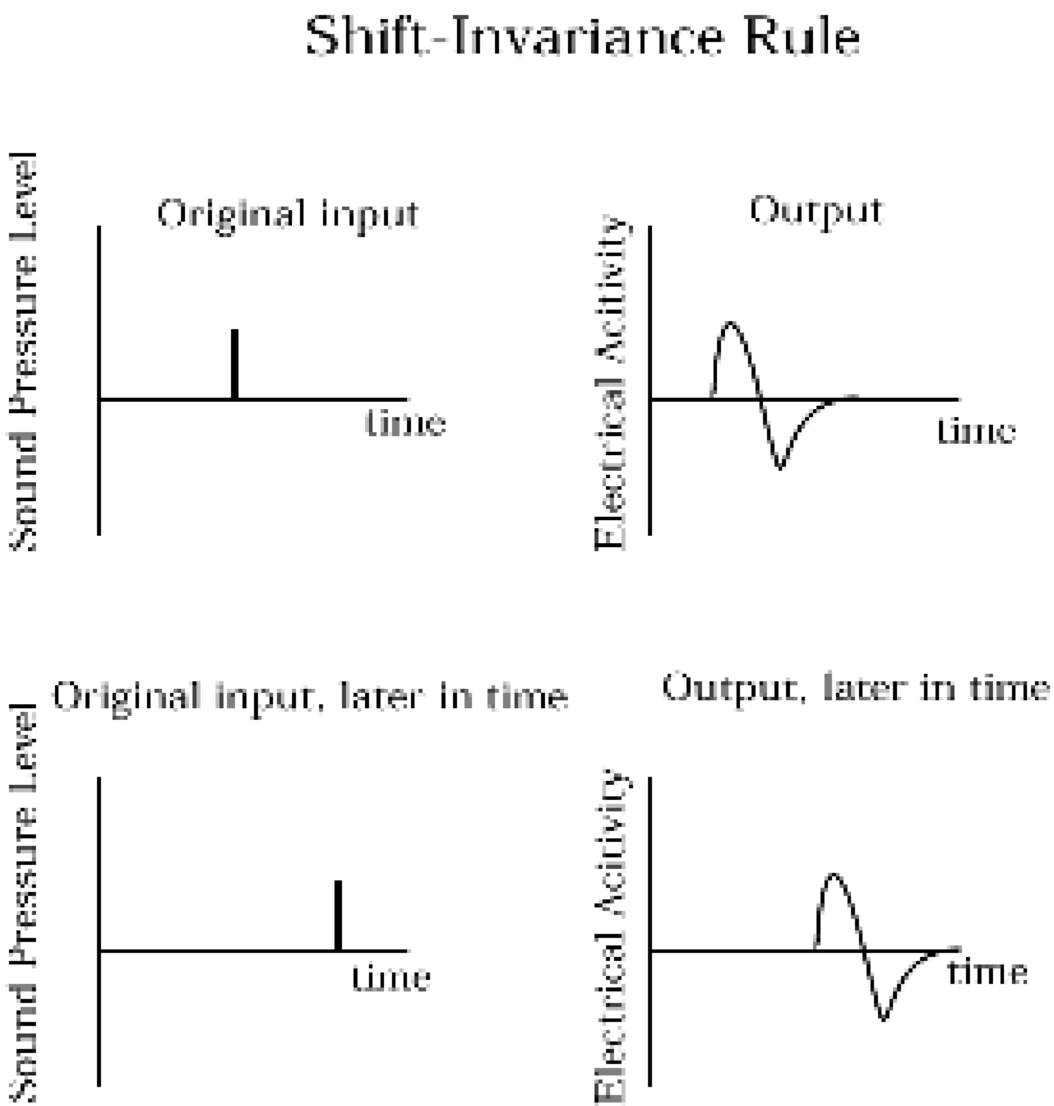
Shift Invariant Systems



Covered in lecture

Shift Invariant Systems (II)

- A system is called ***shift-invariant*** if

$$g(x-\Delta x, y-\Delta y) = S\left[f(x-\Delta x, y-\Delta y)\right]$$

- Shift-invariance does not require or imply linearity

- The **impulse response function** of a shift invariant system is

$$h(x, y, \xi, \eta) = S[\delta_{\xi,\eta}(x, y)] = h(x-\xi, y-\eta)$$

4-D → 2-D

Covered in lecture

Shift Invariant Systems (III)

- The impulse response function of a shift invariant system is

$$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)h(x,y,\xi,\eta)d\xi d\eta$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)h(x-\xi,y-\eta)d\xi d\eta$$
$$=f(x,y)*h(x,y)$$

- The output of a linear and shift-invariant system is the input ***convolved*** with the impulse response function.

Covered in lecture

Linear Systems – Why Important?

- Many imaging systems used in medical and other applications can be described as linear systems.

- Ideally, we should have

Image = Object * Impulse Response Function + Noise

Convolution process

- We are not quite there yet …

Covered in lecture

Impulse Response Function

- The **impulse response function** is defined as

$$h(x,y,\xi,\eta) = S[\delta(x-\xi, y-\eta)]$$

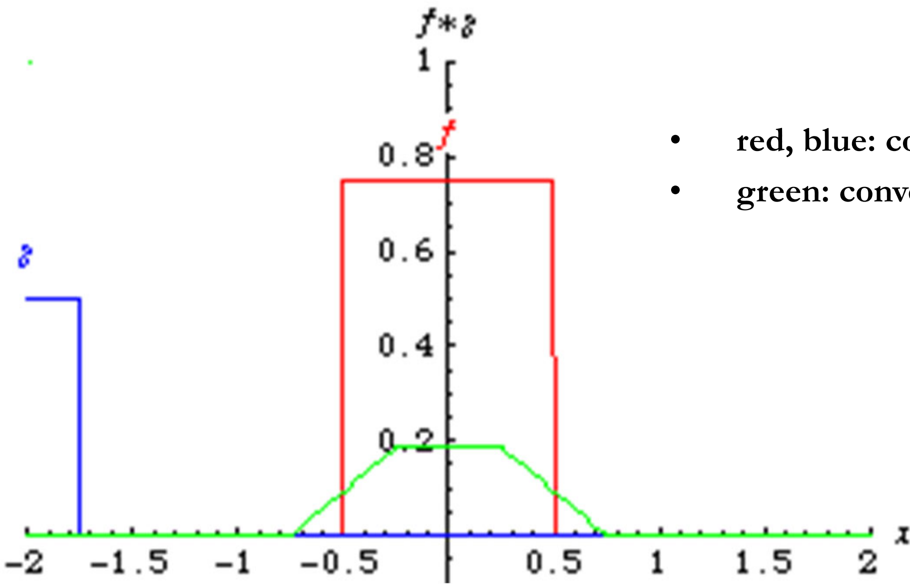
- The impulse response function is sometimes referred to as the **point-spread function** (PSF).

Covered in lecture

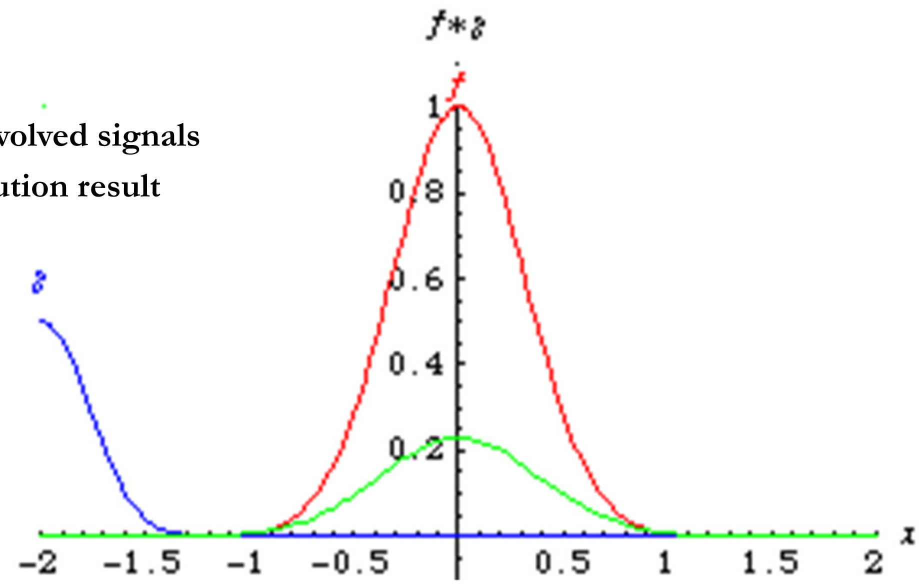
Convolution Operation in 1-D – Examples

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(\xi) g(x - \xi) d\xi$$

- **red, blue: convolved signals**
- **green: convolution result**



two boxes



two Gaussians

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Properties of Convolution Operation

$$f(x,y)*h(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)h(x-\xi,y-\eta)d\xi d\eta$$

- **Commutativity**

$$h_1(x,y)*h_2(x,y)=h_2(x,y)*h_1(x,y)$$

- **Distributivity**

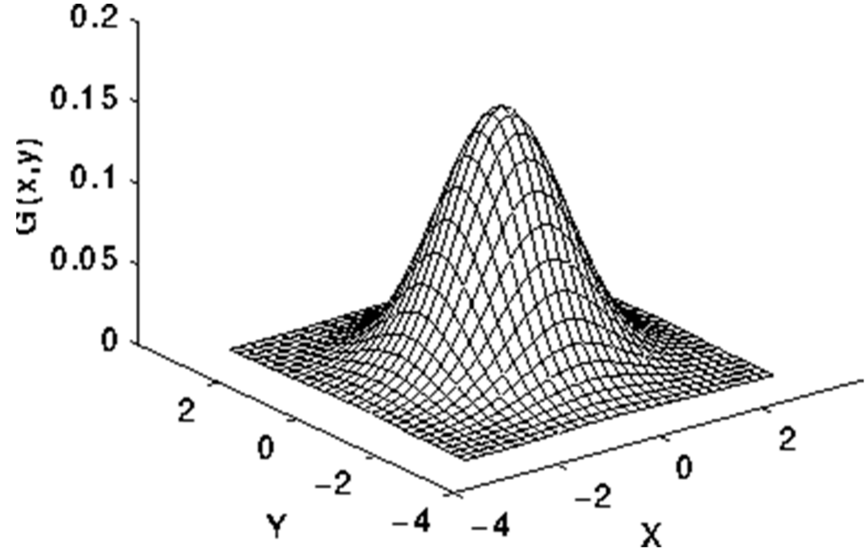
$$\left[h_1(x,y)+h_2(x,y)\right]*f(x,y)=h_1(x,y)*f(x,y)+h_2(x,y)*f(x,y)$$

Convolution Operation – Examples

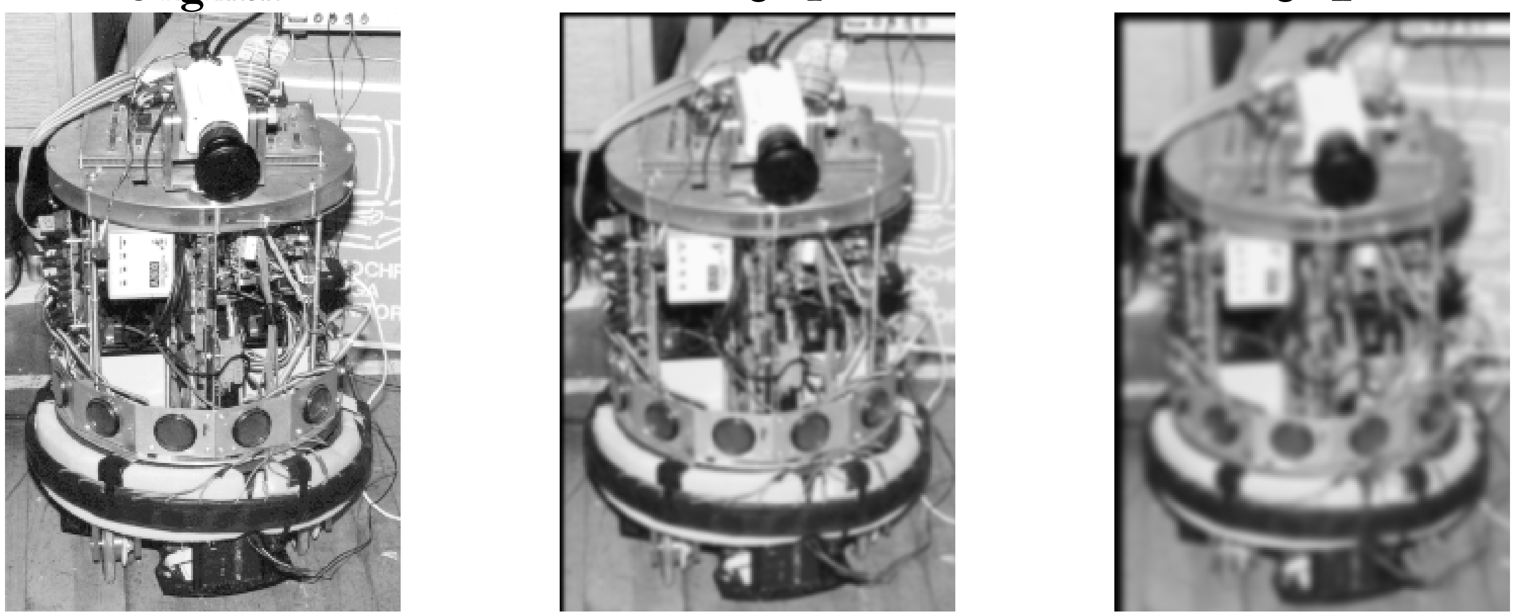
$$g(x,y) = f(x,y) * gaussian(x,y)$$

where

$$gaussian(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$



original **σ=1** **σ=2**



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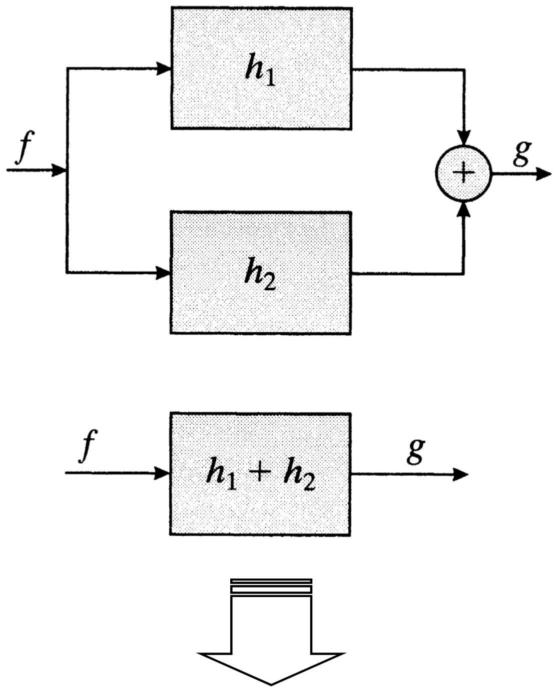
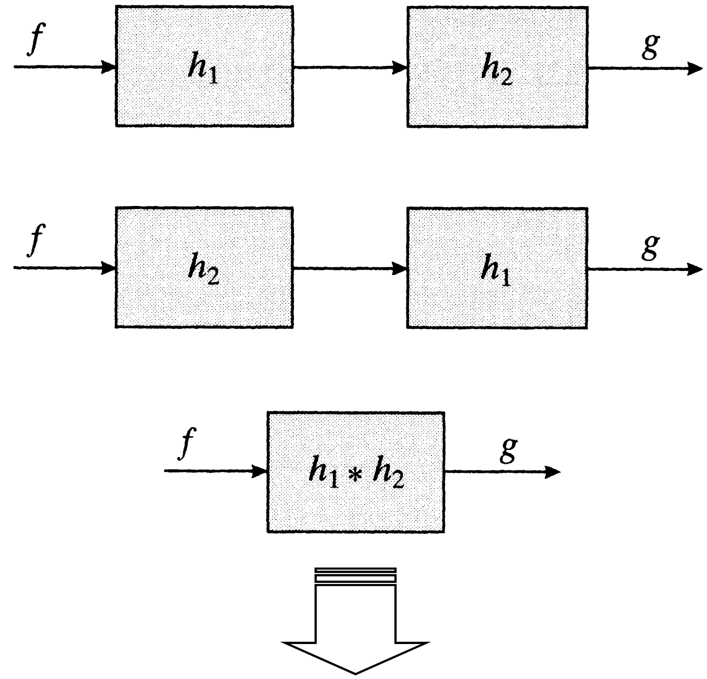
Connection of LSI Systems





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$$g(x,y) = [h_1(x,y) + h_2(x,y)] * f(x,y)$$
$$= [h_2(x,y) + h_1(x,y)] * f(x,y)$$

$$g(x,y) = h_2(x,y) * [h_1(x,y) * f(x,y)]$$
$$= h_1(x,y) * [h_2(x,y) * f(x,y)]$$

LSI systems may be decomposed into the combination of multiple sub-systems. This may lead to a simplified mathematical representation of the complete system…

Separable Systems

A system is called **separable** if

$$h(x, y) = h_1(x)h_2(y)$$

in which case, the convolution between the input and the impulse response function is

Covered in lecture

Separable Systems

For a separable system, the 2-D convolution operation can be re-write as two 1-D convolution operations.

$$g(x,y) = h(x,y) * f(x,y) = h_1(x) * [h_2(y) * f(x,y)]$$

An example

$$gaussian(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}} = \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \right) \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} \right)$$

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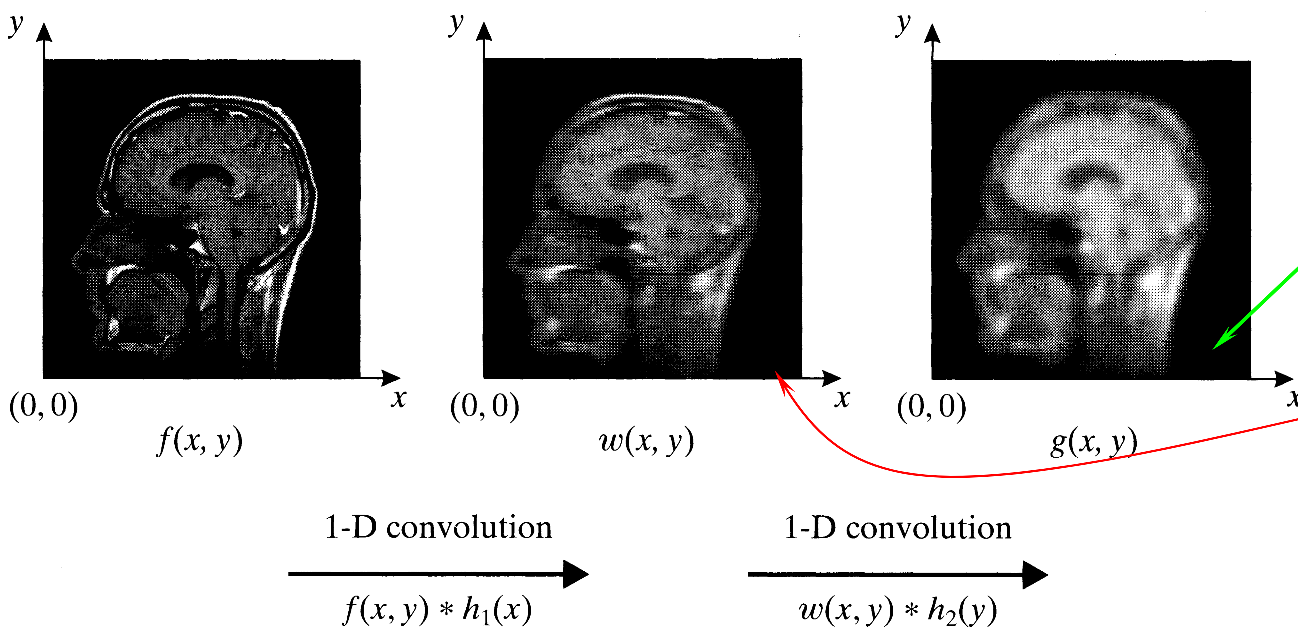
Separable Systems – An Example

An input image pass through a separable system having a impulse response function described by a 2-D Gaussian function

$$gaussian(x,y)=\frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}\cdot\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{y^2}{2\sigma^2}}$$



$$g(x,y)=f(x,y)*gaussian(x,y)$$

$$=f(x,y)*\frac{1}{2\pi\sigma^2}e^{-\frac{x^2+y^2}{2\sigma^2}}$$

$$=\left[f(x,y)*\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}\right]*\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{y^2}{2\sigma^2}}$$

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Summery of Key Concepts

- **Signals** can be described as multi-variate functions.
- Arbitrary signals may be represented (or approximated) by linear combinations of some basic signal functions, such as delta signal, rect signal etc.
- The **impulse response function** of a given system is the output from an delta input signal.
- A system is **linear** if when the input consists of a collection of signals, the output is a summation of the responses of the system to each individual input signal.
- A system is **shift-invariant** if an arbitrary translation of the input results in an identical translation of the output.
- A linear and shift-invariant (**LSI**) system may be described as a **convolution** operator.

Covered in lecture