Chapter 2: Mathematical Preliminaries

Image Quality

What makes for a Good Quality Image?

Sharpness, Contrast, Low noise …

Purpose of the Lecture

- Applies to all types of images
- 'Quality': subjective notion, dependent on image function
- Bottom line outcome measure of a radiological image is its ***usefulness in determining an accurate diagnosis***
- Understanding the image characteristics that comprise image quality is important so that radiologists can recognize problems and articulate their cause
- Introduction to the terminology used for various metrics by physicists and engineers to measure image quality, e.g., contrast, spatial resolution and noise

Covered in lecture

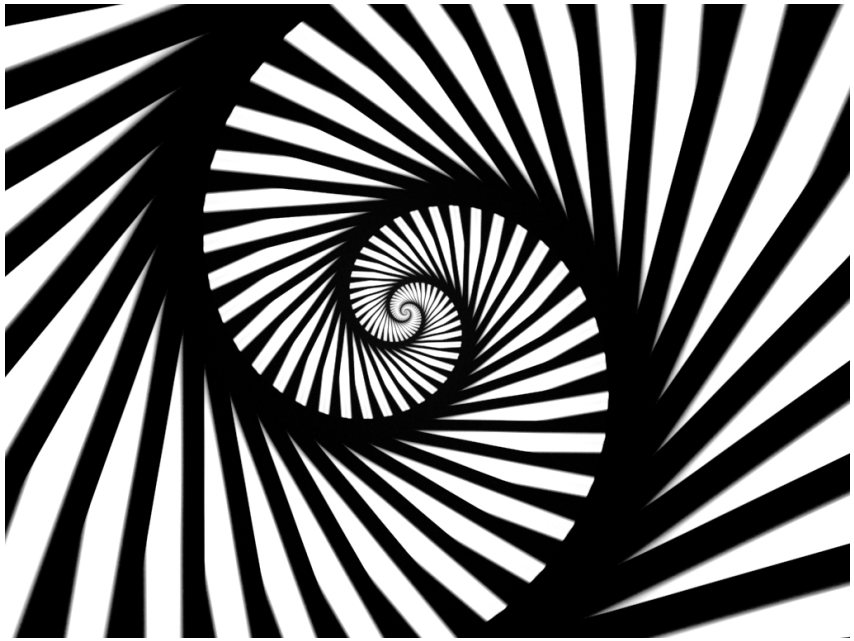
A Revisit to Key Image Quality Measures

Contrast

- What is contrast of an image?
  How to quantify contrast?
  How an imaging system affect contrast?

Covered in lecture

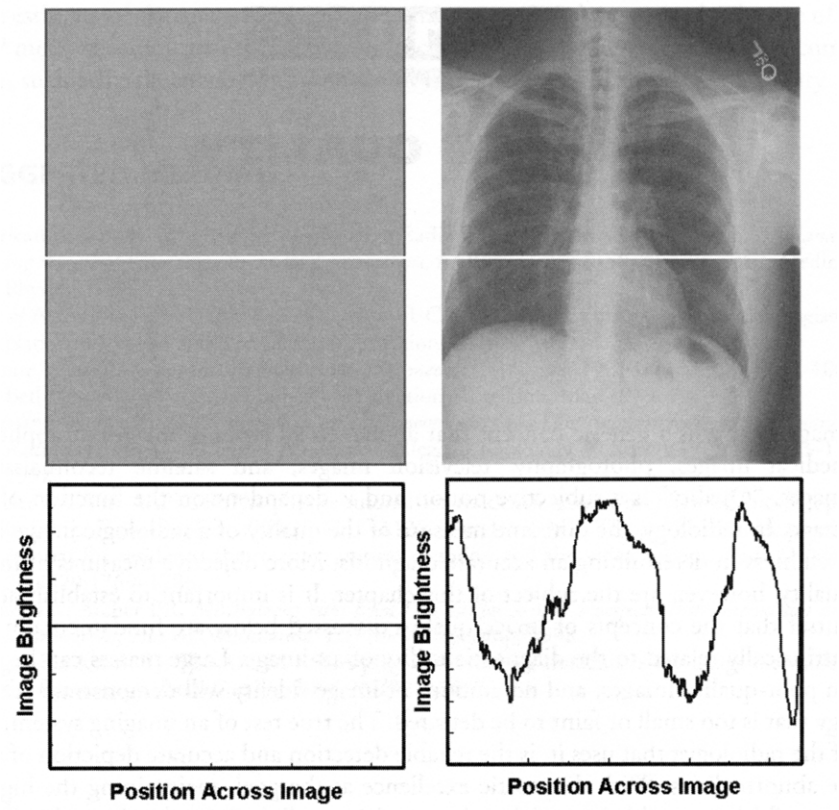
Contrast



Covered in lecture

Contrast

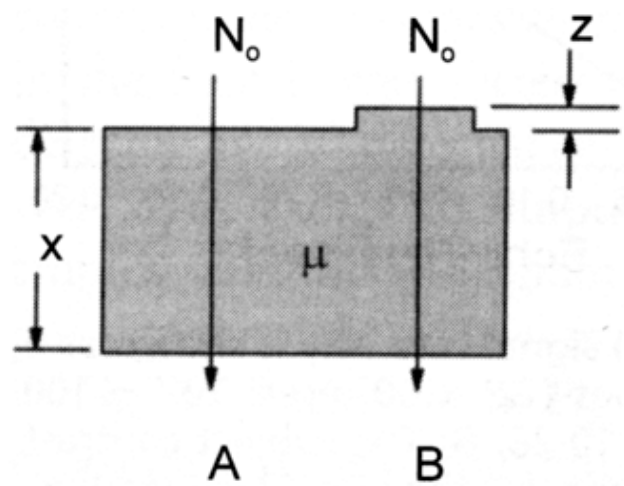
- What is contrast?
- **The difference in the image gray scale** between closely **adjacent regions** of the image
- Medical image contrast is the result of many steps during image acquisition, processing and display

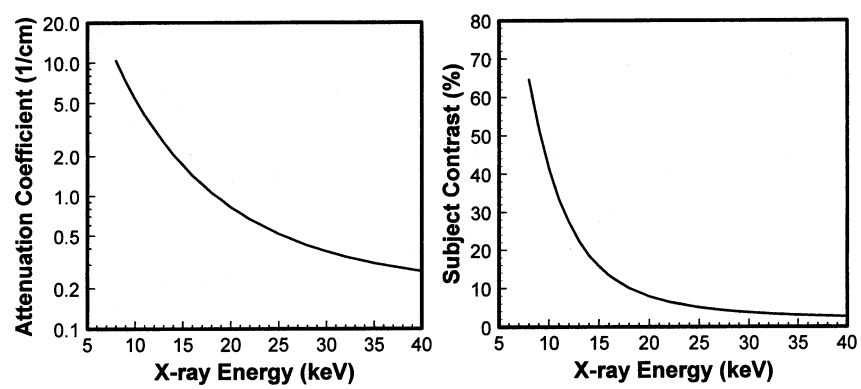


Covered in lecture

Subject (or Object) Contrast

- Difference in some aspects of the signal prior to it being recorded
- Consequence of fundamental differences in the object, e.g., in x-ray intensity based on attenuation
- C = (A-B)/A

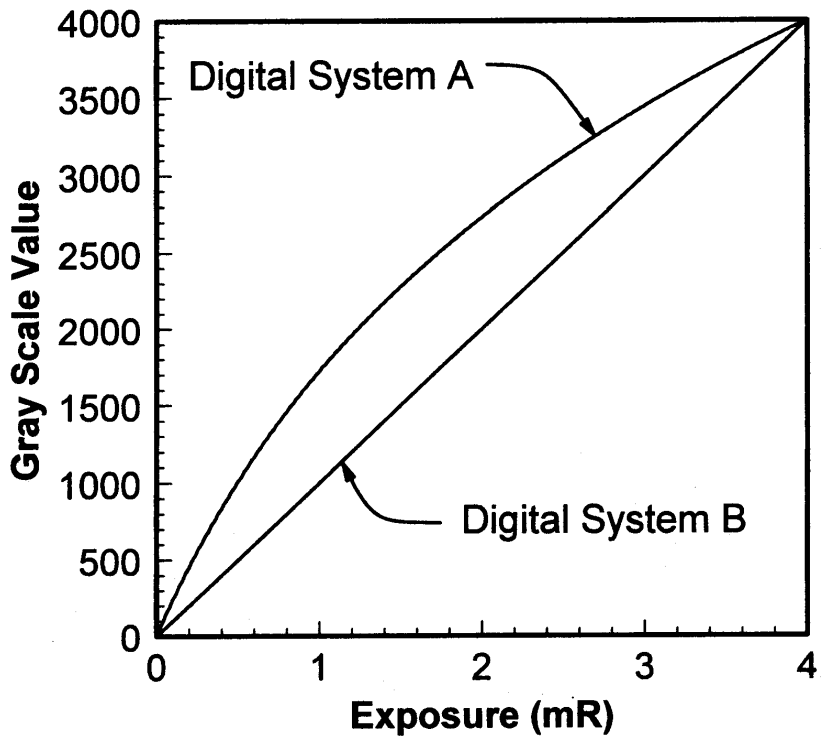




Covered in lecture

Detector Contrast

- Detector Contrast (C_d)
- A detector's characteristics play an important role in producing contrast in the final image
- C_d determined principally by how the detector 'maps' detected energy into the output signal
- Characteristic curve: input radiation exposure to output value (analog or digital)



Covered in lecture

A Revisit to Key Image Quality Measures

Modulation

- What is modulation?
- Definition of modulation transfer function
- How to experimentally measure MTF

Covered in lecture

Modulation

Recall that **an arbitrary signal can be decomposed into the weighted sum of periodic signals**, such as sinusoidal waves, so study the response of an imaging system to such signals provide a effective approach for quantifying the image quality.

Covered in lecture

Modulation

- The modulation m_f is an effective way to quantify the contrast of a periodic signal

$$m_f = \frac{f_{\max} - f_{\min}}{f_{\max} + f_{\min}}.$$

- In general, m_f is refer to as ***the contrast of a periodic signal f(x,y) relative to its average value***.

- So within two signals, f(x,y) and g(x,y), with the same average value , f(x,y) is said to have ***more contrast*** if $m_f > m_g$.

Covered in lecture

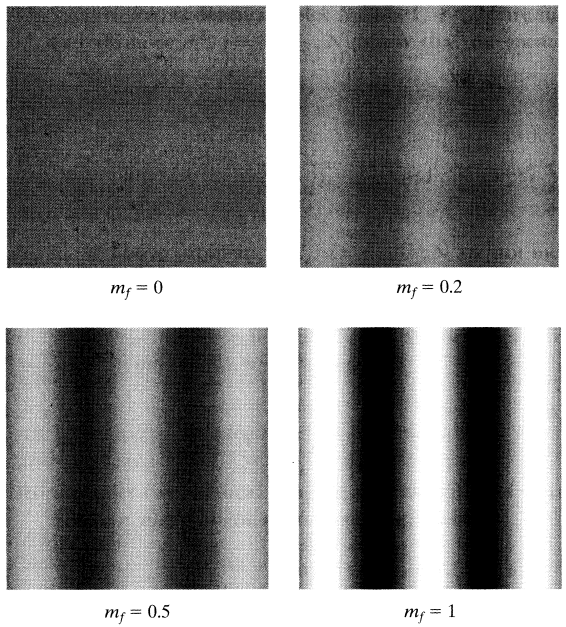
Modulation

- Suppose an input signal function

$$f(x, y) = A + B\sin(2\pi u_0 x)\,,$$

where A>B and both are non-negative constants.

$$m_f = \frac{f_{\max} - f_{\min}}{f_{\max} + f_{\min}}\,. \quad \Rightarrow \quad m_f = \frac{B}{A}\,.$$



Greater m_f, more contrast

Covered in lecture

Modulation

- Now let this signal to pass through an LSI imaging system. Suppose an input signal function. Since

$$f(x,y) = A + B\sin(2\pi u_0 x) = A + \frac{B}{2j}\left[e^{j2\pi u_0 x} - e^{-j2\pi u_0 x}\right],$$

- Suppose the system impulse response function h(x,y) is circularly symmetric,

$$g(x,y) = AH(0,0) + B\,|H(u_0,0)|\sin(2\pi u_0 x)\,.$$

- So the modulation of the output signal is

$$m_g = \frac{B|H(u_0,0)|}{AH(0,0)} = m_f\frac{|H(u,0)|}{H(0,0)}\,.$$

Covered in lecture

Modulation

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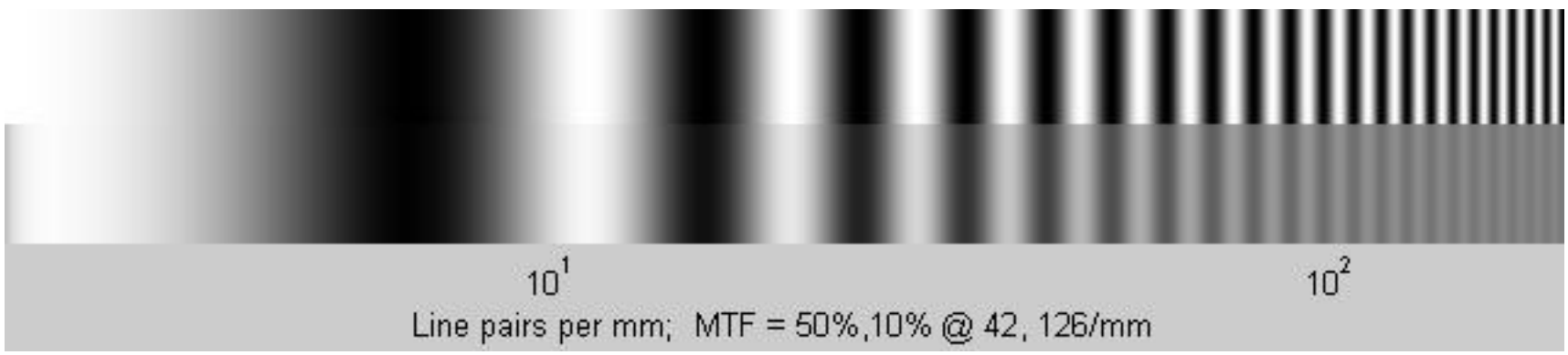

Covered in lecture

Modulation

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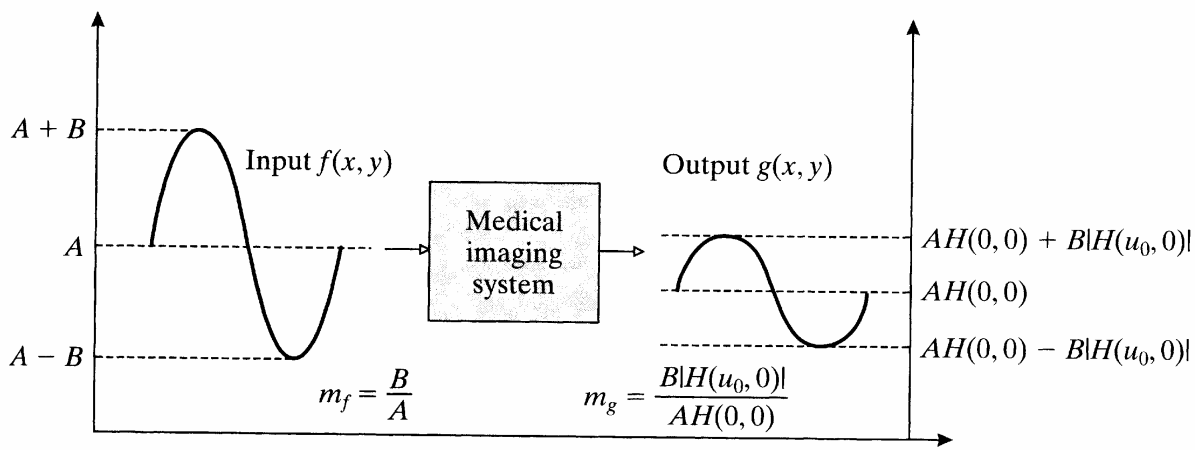
Covered in lecture

Modulation

- The effect of an LSI having circular symmetric impulse response function on an input sinusoidal signal is to ***scale the input signal by a factor equal to the magnitude spectrum at the same frequency u_0.***



- It is often true that H(0,0) ≅1, and $H(U_0,0)$ <1. So that the output signal has less contrast, $m_g < m_f$.

Covered in lecture

Modulation Transfer Function (MTF)

- In order to fully quantify the response of an LSI for an arbitrary signal, f(x,y), we would need to know the response of the system to sinusoidal signals at different frequencies.

- Modulation Transfer Function (MTF).

$$\mathrm{MTF}(u) = \frac{m_g}{m_f} = \frac{|H(u,0)|}{H(0,0)} .$$

- MFT is, in effect, the "frequency response function" of an given imaging system. It is normally evaluated for positive frequencies only.

- Most imaging systems lead to decreased contrast, so that

$$0 \leq \mathrm{MTF}(u) \leq \mathrm{MTF}(0) = 1, \qquad \text{for every } u ,$$

Covered in lecture

Modulation Transfer Function (MTF)

- In case of non-isotropic impulse response function (h(x,y) is not circularly symmetric), the MTF can be defined as

$$\mathrm{MTF}(u,v) = \frac{m_g}{m_f} = \frac{|H(u,v)|}{H(0,0)},$$

- A typical MTF of an imaging system

$$|H(u,v)| = \sqrt{H_R^2(u,v) + H_I^2(u,v)}$$

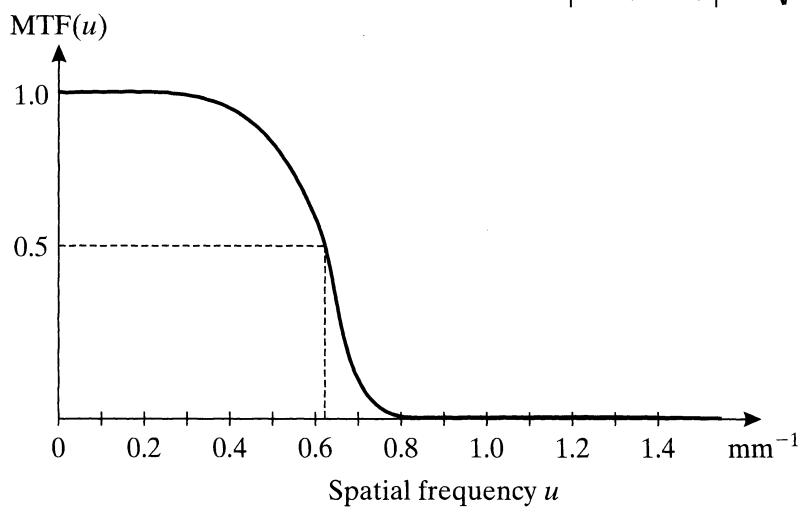


Figure 3.3
A typical MTF of a medical imaging system.

Modulation Transfer Function (MTF)

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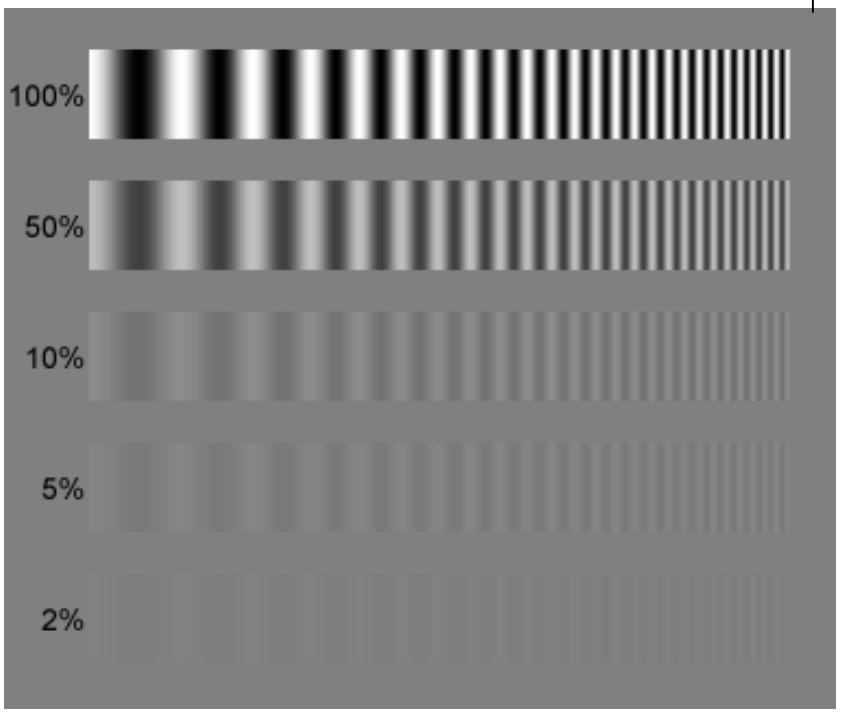

Uniform MTF

Covered in lecture

Modulation Transfer Function (MTF)

$$\mathrm{MTF}(u,v) = \frac{m_g}{m_f} = \frac{|H(u,v)|}{H(0,0)},$$

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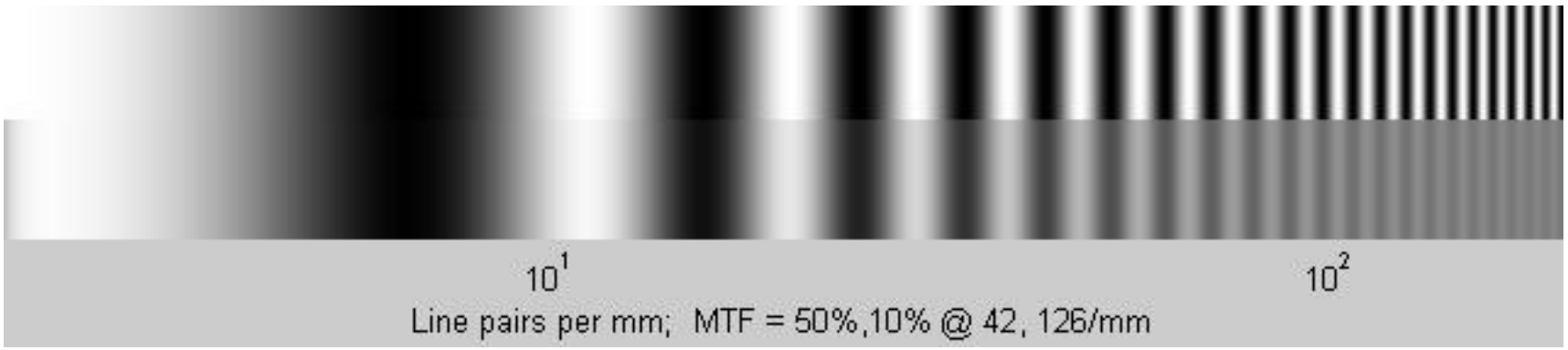
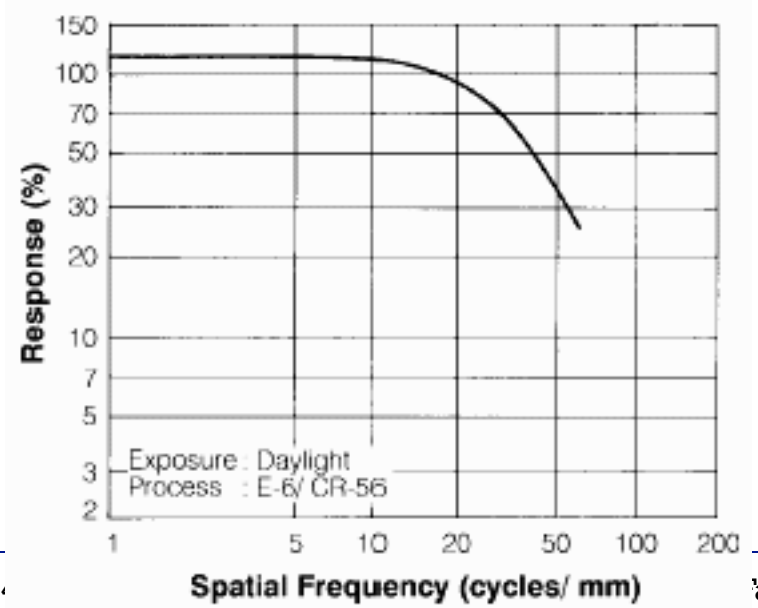




Covered in lecture

System Cascade

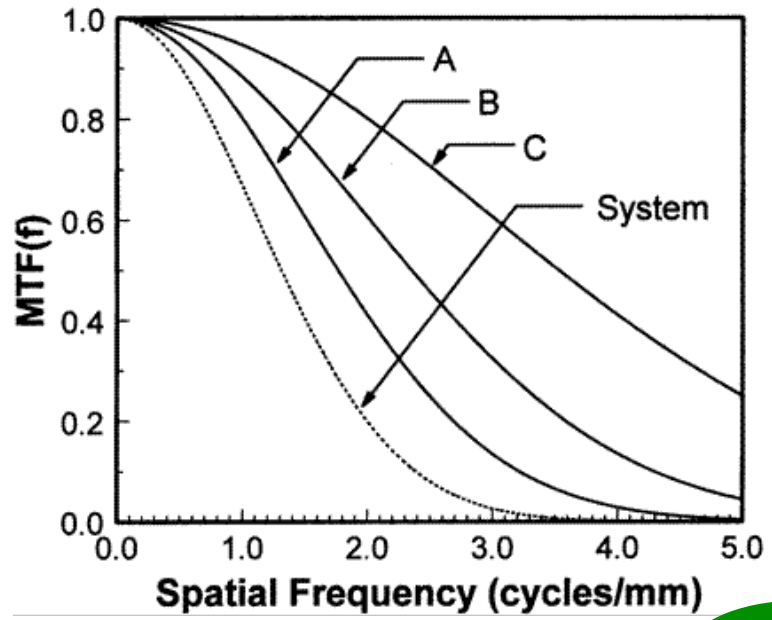
- Modulation Transfer Function (MTF).

$$g(x,y) = h_K(x,y) * \cdots * (h_2(x,y) * (h_1(x,y) * f(x,y))) .$$

- The overall MTF is the product of the MTF for sub-systems:

$$MTF_{total}(u) = \prod_{i=1}^{i=I} MTF_i(u)$$

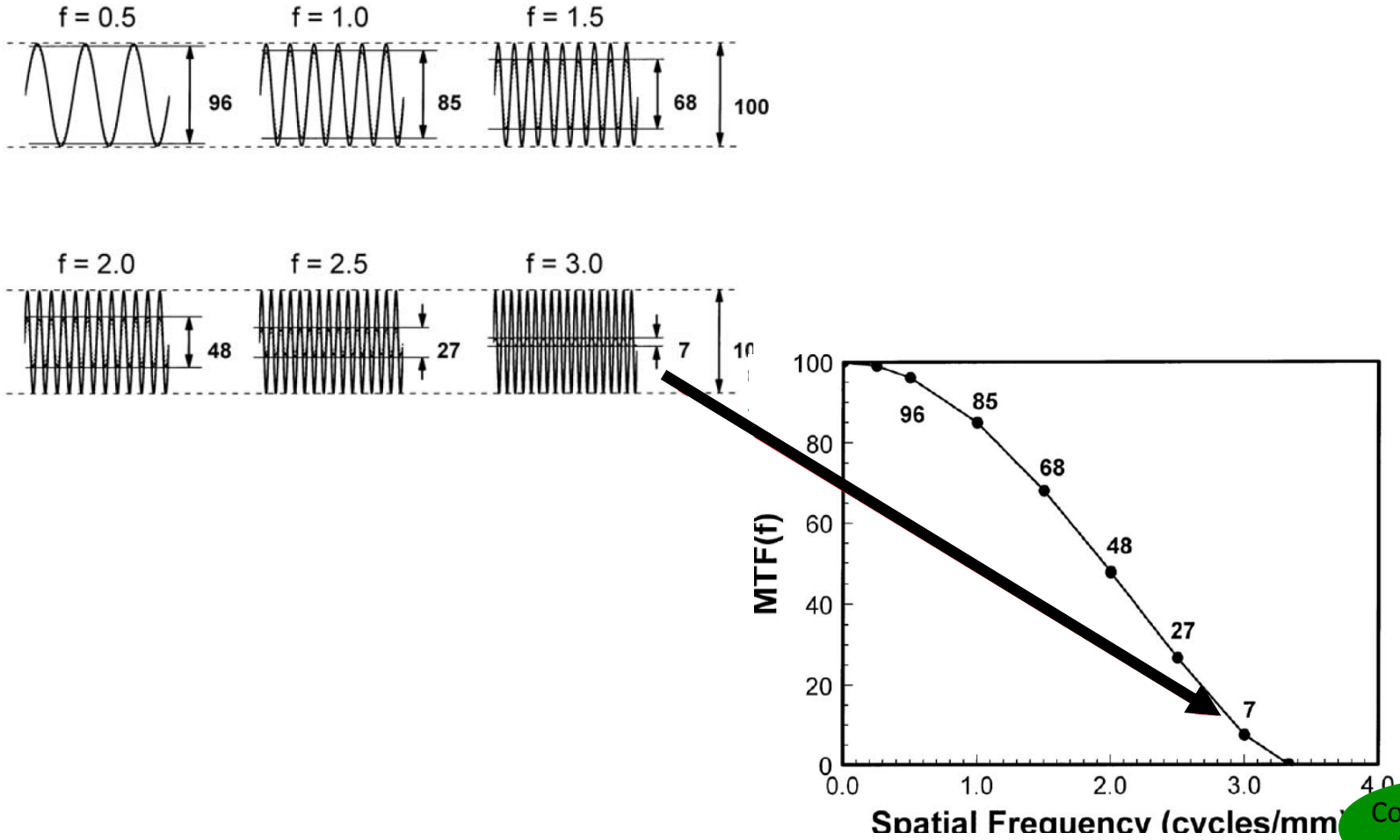
- MTF of an imaging system that can be modeled as a chain if systems is often determined by the MTF of the worst system in the cascade.



Covered in lecture

Modulation transfer Function -- Revisited

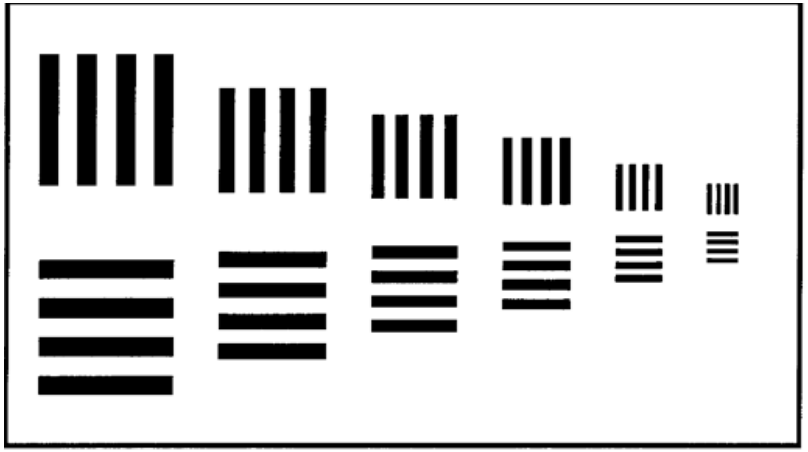
- Modulation Transfer Function (MTF).

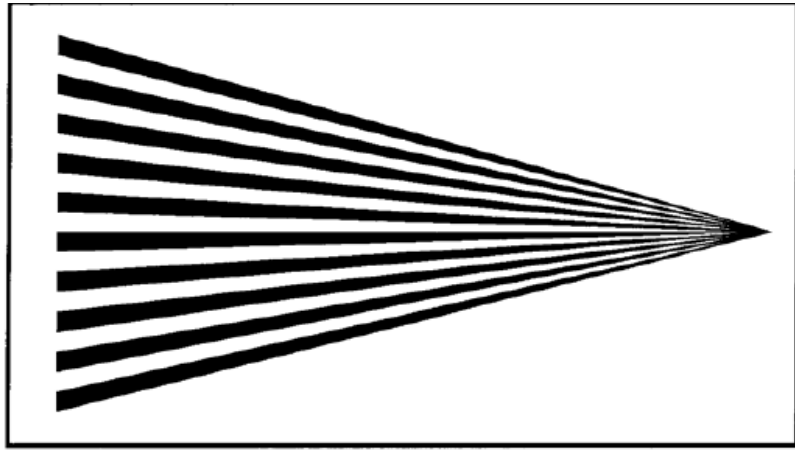


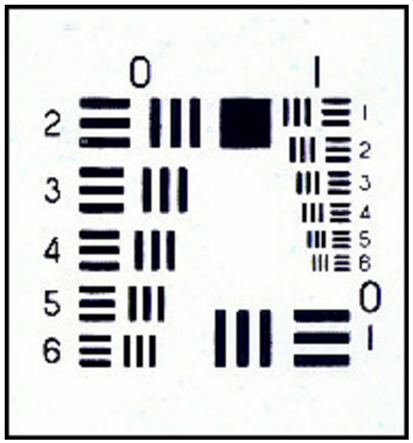
Covered in lecture

Modulation transfer Function -- Revisited

Standard Test Chart for Determining the Modulation Transfer Function (MTF) of an Imaging System.

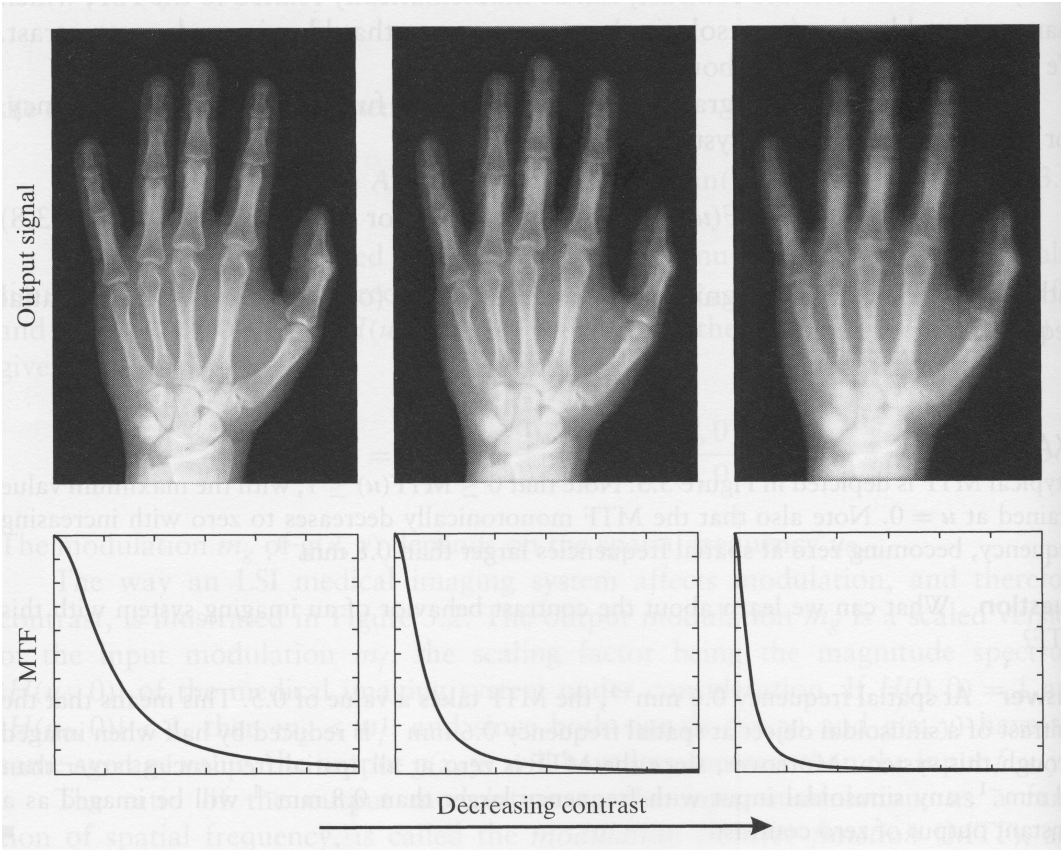






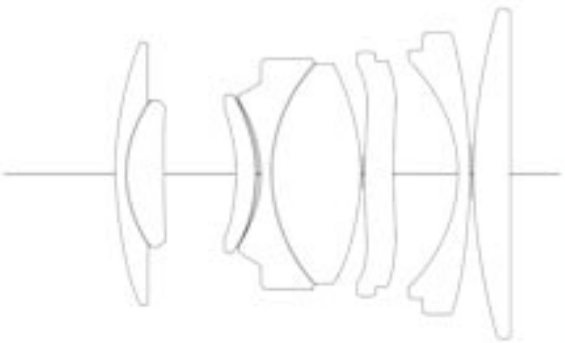
Modulation Transfer Function (MTF)

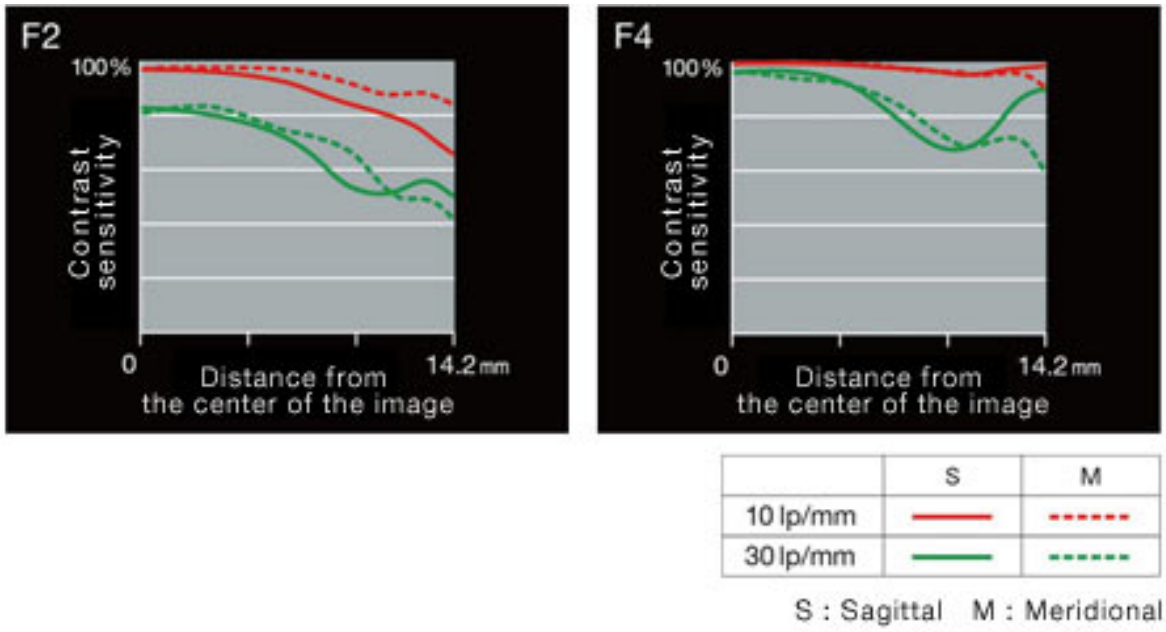
- An example of the same signal passing through three imaging systems with decreasingly poor MTF, which leads to decreasing contrast in images



Covered in lecture

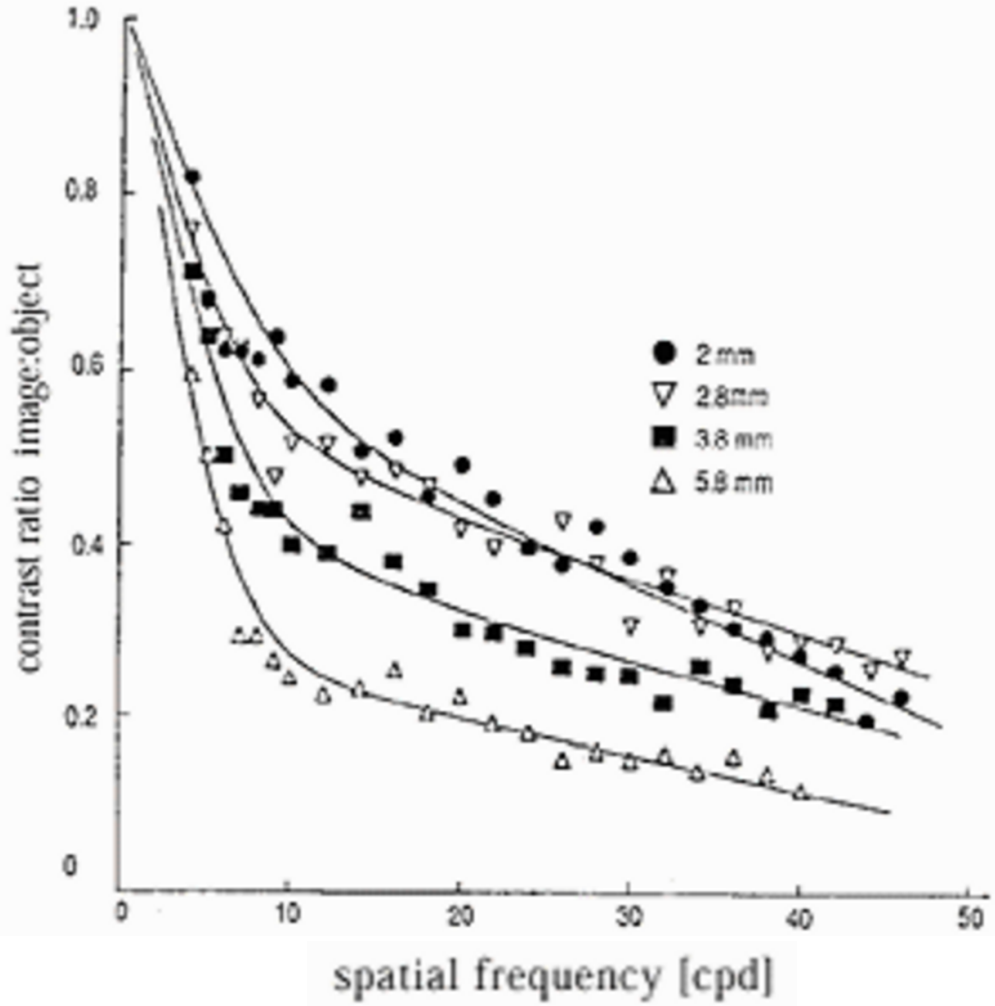
Modulation Transfer Function (MTF)





Optical modulation transfer function (MTF) of the human eye

- MTF is measured directly with sinewave gratings.
- The optical modulation transfer function (MTF) can be interpreted as Fourier transform of the optical LSF.



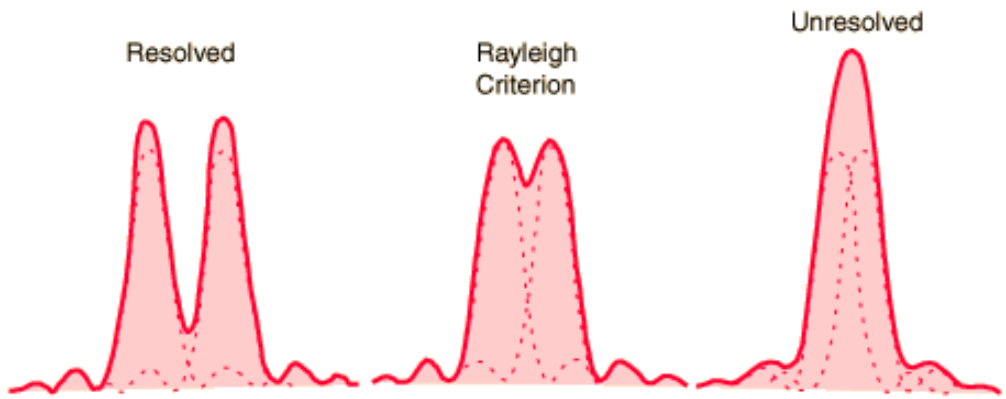
A Revisit to Key Image Quality Measures

Spatial Resolution

- How to quantify spatial resolution?
- How to measure spatial resolution?
- The relationship between spatial resolution and modulation transfer function (MTF)?

Spatial Resolution

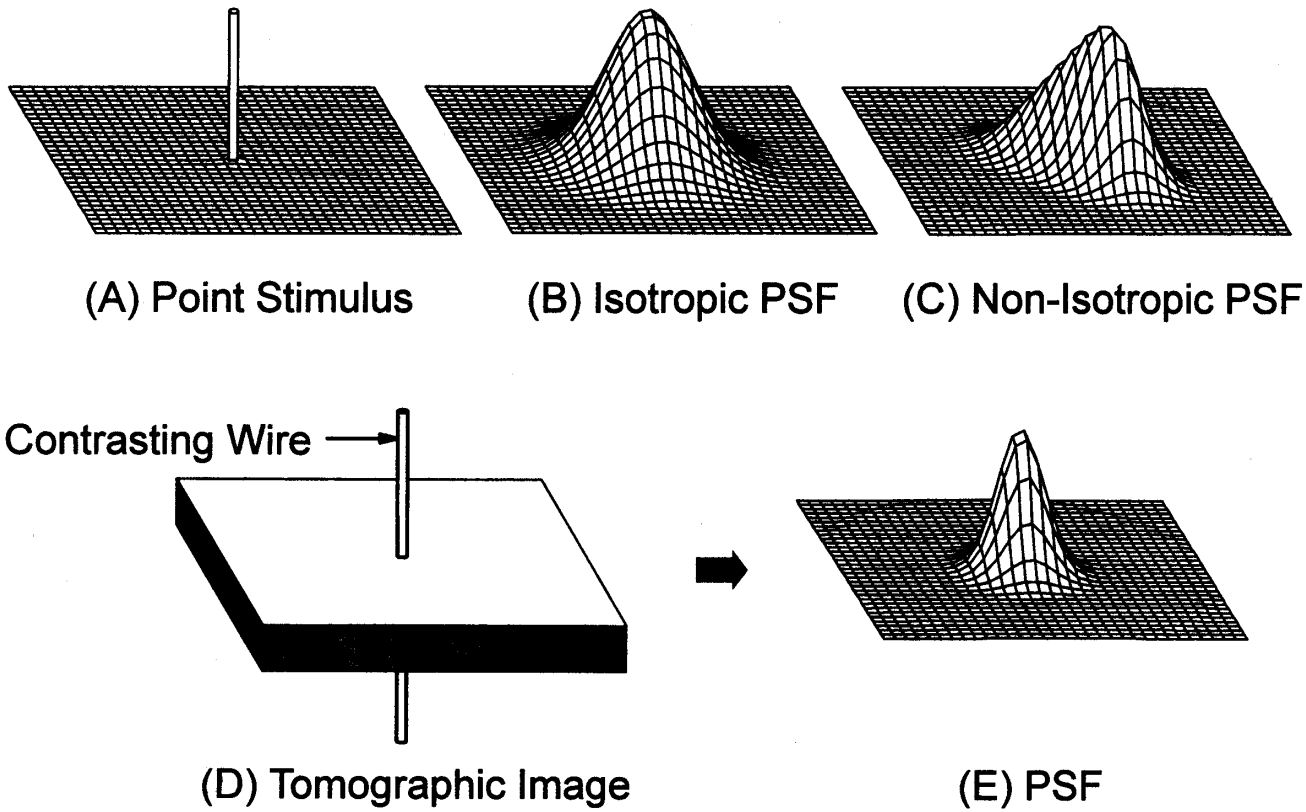
- ***Resolution***: the ability of an given imaging system to ***accurately depict two distinct events in space, time or frequency*** respectively.
- Resolution can also be thought as the ***degree of blurring, smearing***.



- **Spatial resolution** *is fully described by the* ***point-spread function*** or ***impulse response function, h(x,y)***.

The Point Spread Function (PSF)

- One method is to stimulate the imaging system with a point impulse and observe the resulting ***point-response function***.



Full-width at Half Maximum (FWHM) of The Point Spread Function (PSF)

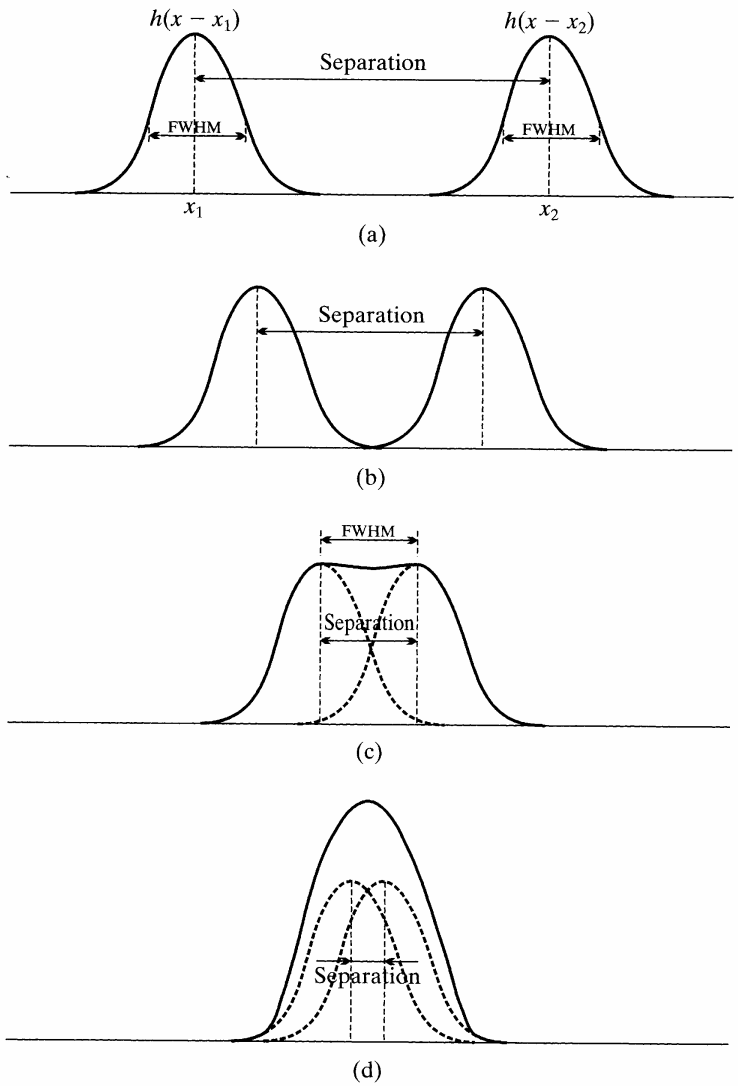
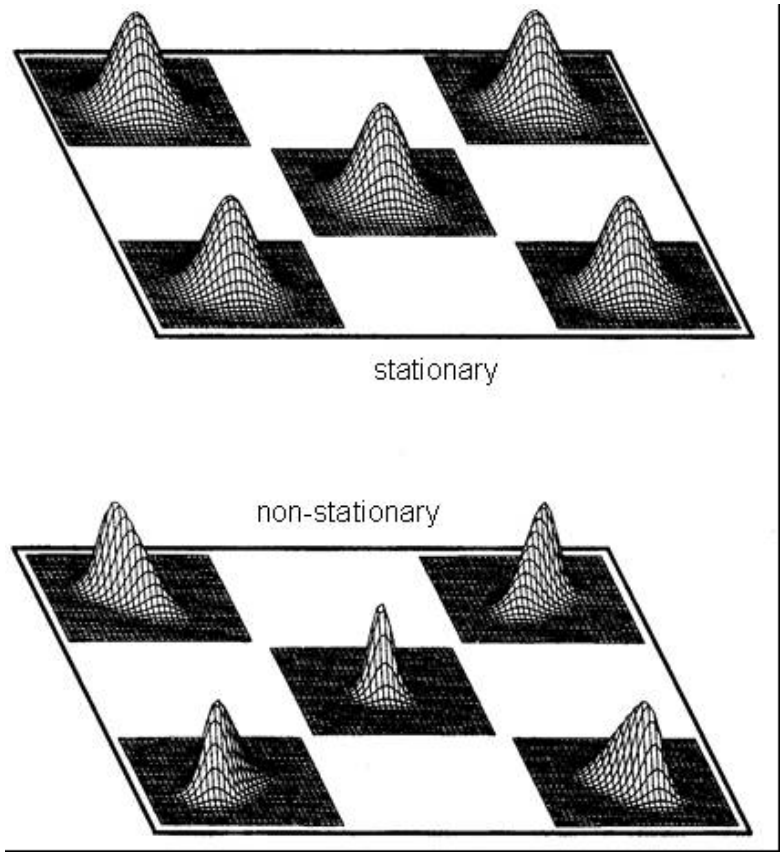


Figure 3.6
An example of the effect of system resolution on the ability to differentiate two points. The FWHM equals the minimum distance that the two points must be separated in order to be distinguishable.

Stationary and Non-stationary PSF

- Spatial variation of the PSF is another important aspect of an given imaging system.



Other Ways to Measure the Spatial Resolution Line Response Function

- The resolution of an imaging system can also be estimated with the ***line-spread function***
- Given a line impulse

$$\textit{line impulse}\ f(x,y) = \delta_\ell(x,y) \ = \delta(x\cos\theta + y\sin\theta - \ell)$$

- Assuming the impulse response function of the imaging system h(x,y) is isotropic, ***line response function*** is

$$\begin{aligned} g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi,\eta) f(x-\xi, y-\eta)\, d\xi\, d\eta\,, \\ &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h(\xi,\eta)\delta(x-\xi)\, d\xi\right] d\eta\,, \\ &= \int_{-\infty}^{\infty} h(x,\eta)\, d\eta\,, \end{aligned}$$

Line Response Function

- The 1-D Fourier transform of the line spread function is

$$
\begin{aligned}
L(u) &= \mathcal{F}_{1D}[l](u)\,, \\
&= \int_{-\infty}^{\infty} l(x) e^{-j2\pi ux}\, dx\ , \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,\eta) e^{-j2\pi ux}\, dx\, d\eta\,, \\
&= H(u,0).
\end{aligned}
$$

- So the values of the Fourier transform of the LSF crossing an horizontal line passing through the origin is sufficient for describing the LSF

LSF and MTF

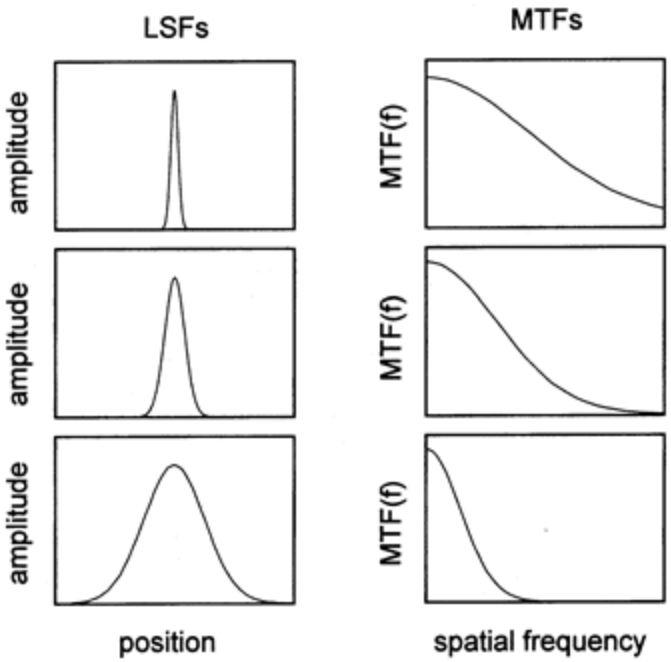
- Modulation Transfer Function (MTF).

$$\mathrm{MTF}(u) = \frac{m_g}{m_f} = \frac{|H(u,0)|}{H(0,0)} = \frac{|L(u)|}{L(0)}, \qquad \text{for every } u.$$

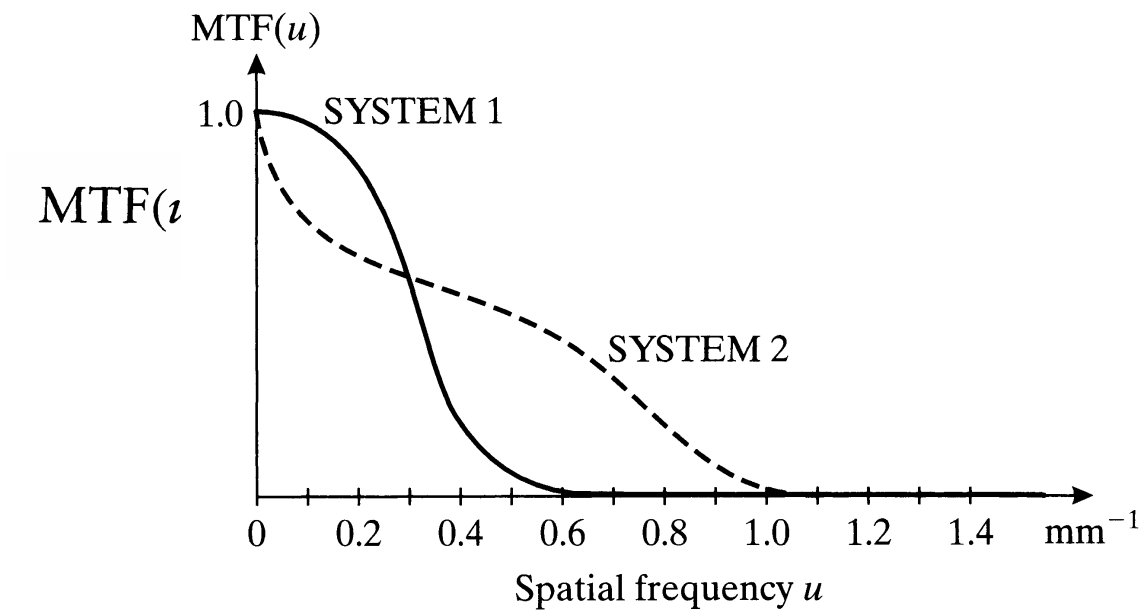
- For a "reasonable" imaging system, the L(0)=1. so that

$$\mathrm{MTF}(u) = L(u)$$

- MTF is an effective way to compare two imaging systems in terms of spatial resolution and contrast.



LSF and MTF



- System 1 has better low frequency contrast and it is better for imaging coarse features.
- System has a better high frequency contrast. It is better for resolving fine details.
- The resolution of a system is related to the higher frequency components and the cut-off frequency of the MFT.

Image Noise

- The random fluctuation is referred to as the image ***noise***. That has dramatic effects on the subsequent analysis, for example, for signal detection and quantification tasks …

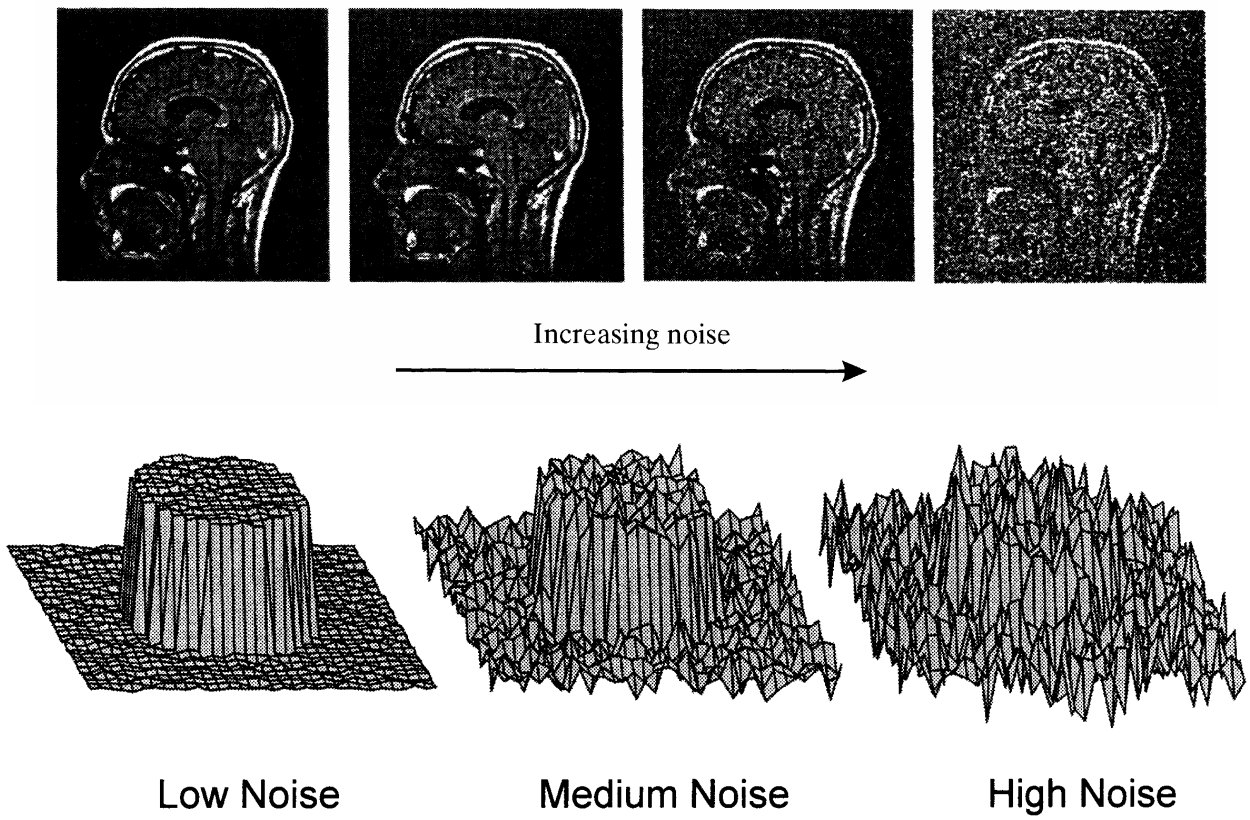
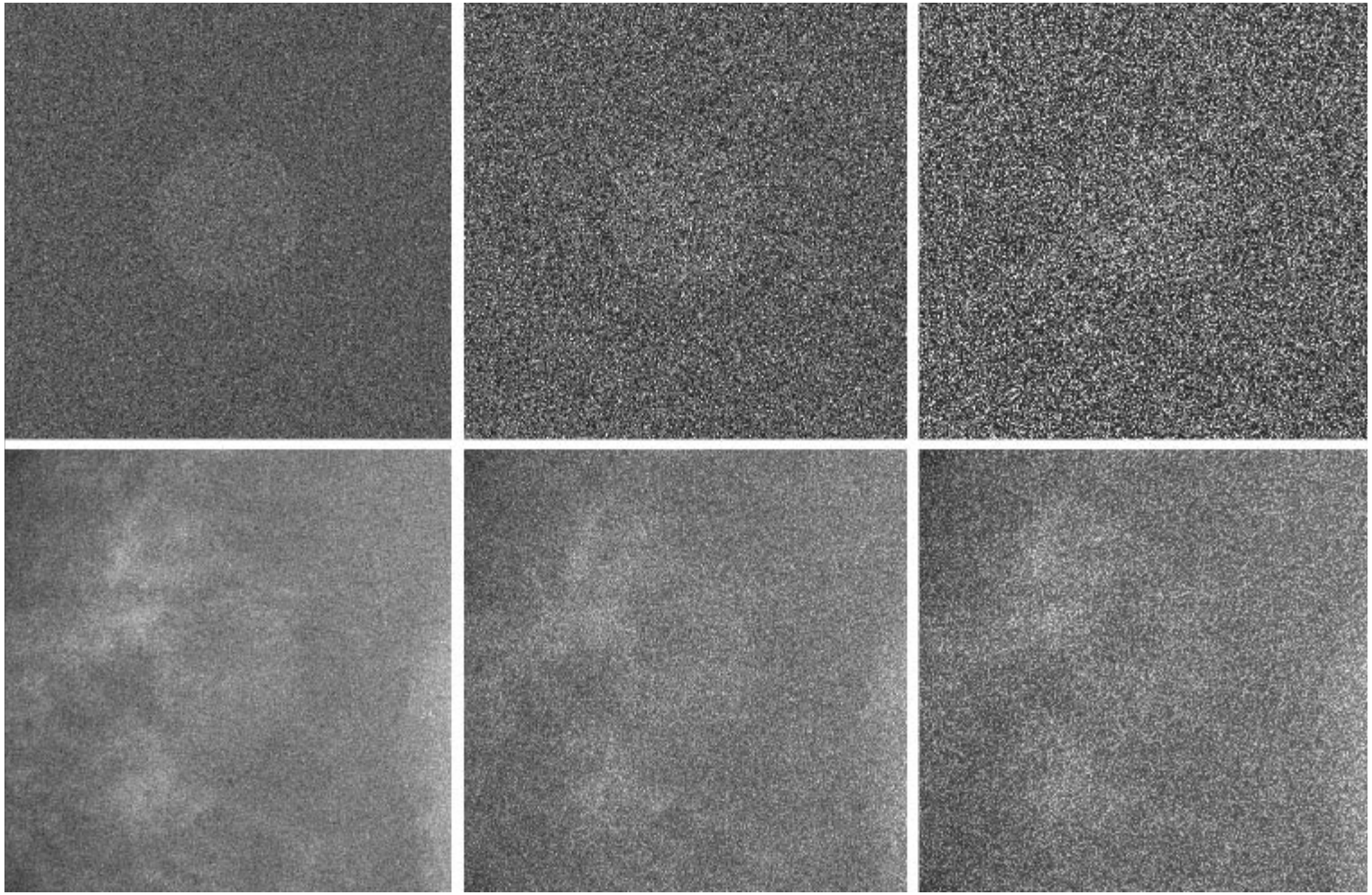


Image Noise Reduces Contrast



Where is the noise coming from?

Image Noise

- Due to the random nature of the data acquired by any imaging system, the output images are normally multivariate random variables.

Input signal ⇨ Data acquisition hardware ⇨ Noisy Data ⇨ Data processing ⇨ Final image

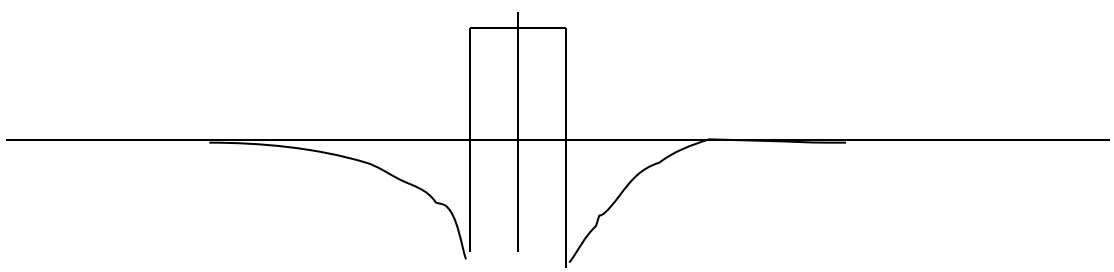
An Example – Noise on Images Acquired with FBP

The Ram-Lak filter in spatial domain

$$h_{\mathrm{RL}}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{\mathrm{RL}}(\omega)\exp(ix\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} |\omega|\,\exp(ix\omega)\,d\omega$$

$$= 2B^2\,\mathrm{sinc}(2\pi Bx) - B^2\,\mathrm{sinc}^2(\pi Bx)$$



$$\hat{f}(x,y) = \frac{1}{\pi}\int_0^{\pi} d\phi \int_{-\infty}^{\infty} dx'\, p_\phi(x')\,h(x\cos\phi + y\sin\phi - x')$$

Noise in Final Image – An Example

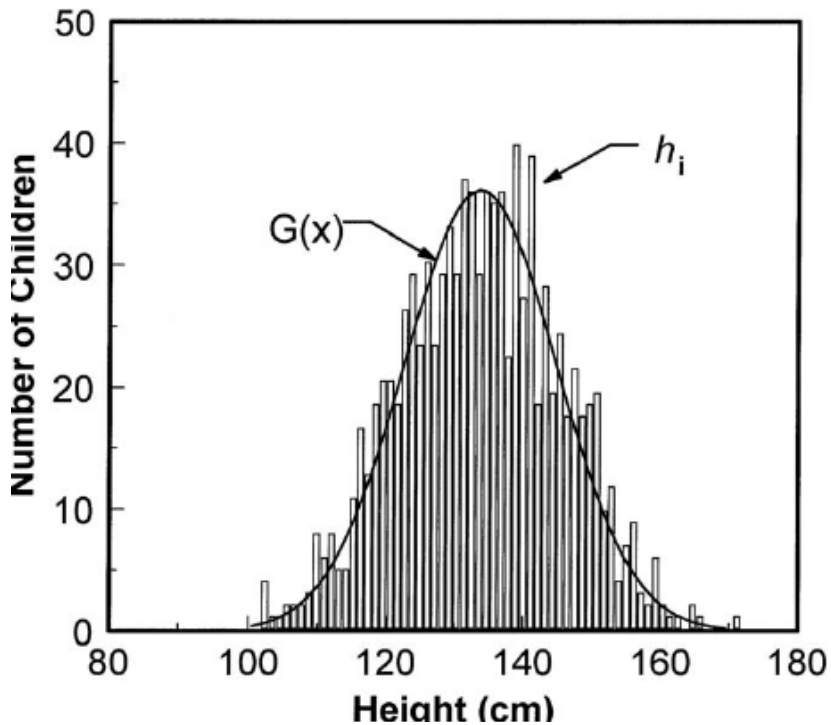
A few observations:

- The noise (**r**) on the final image is a direct consequence of the noise on the projection data.
- Every element in the final image is given as

$$\hat{f}_i = \sum_{j=1}^{M} q_{ij} g_j$$

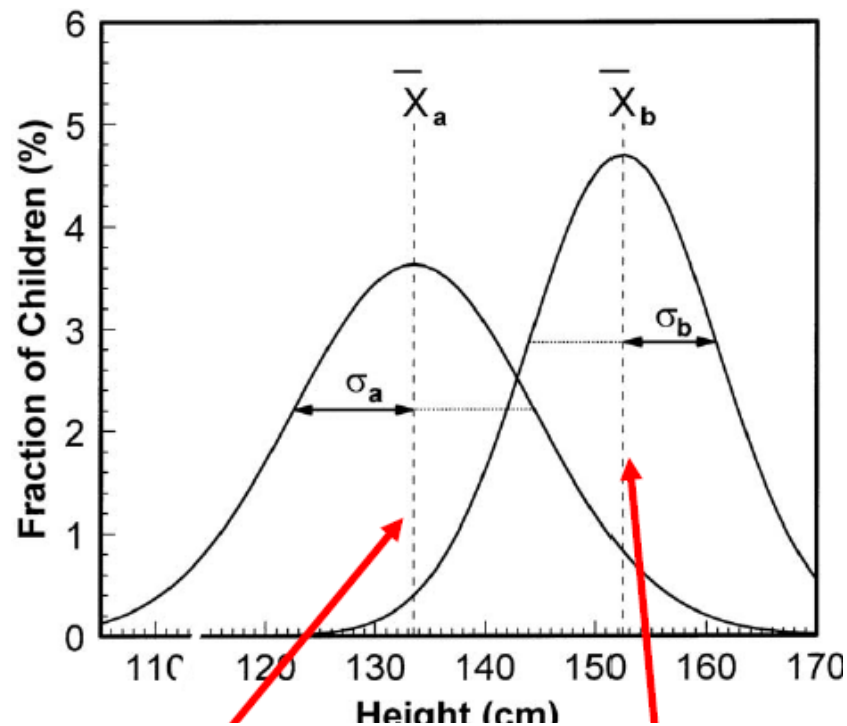
What is the nature of the noise (**r**) ?

Noise on Measured Data -- An Example



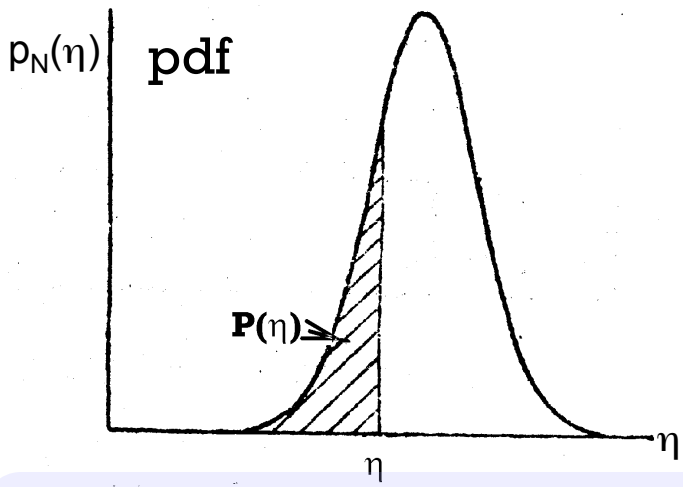
1000 3rd grade heights,

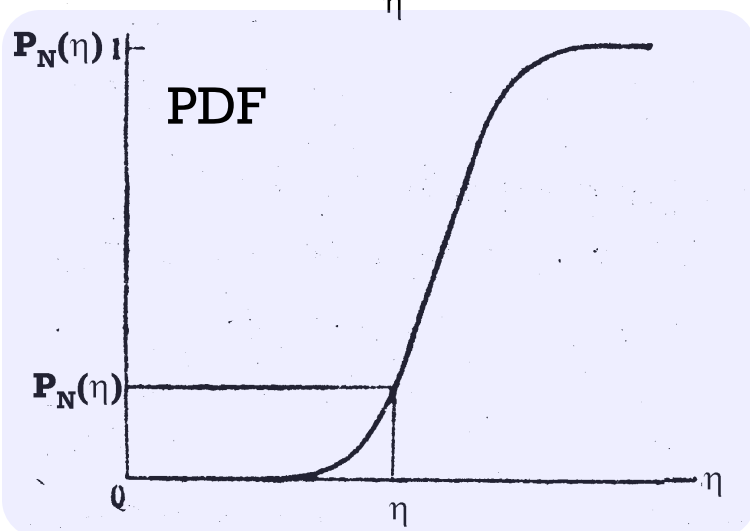
Mean=133.5cm



3rd grade vs. 6th grade heights

Probability Distribution Function for Random Variables





- For a random variable, the so-called ***probability distribution function*** is defined (PDF) as

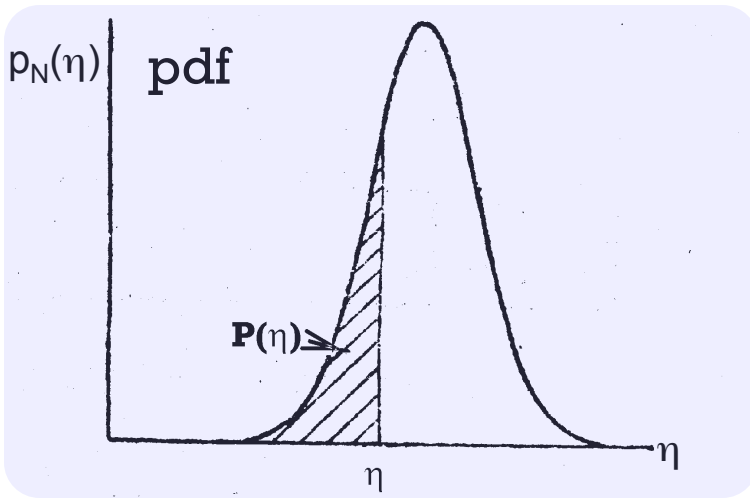
$$P_N(\eta) = \Pr[N \leq \eta],$$

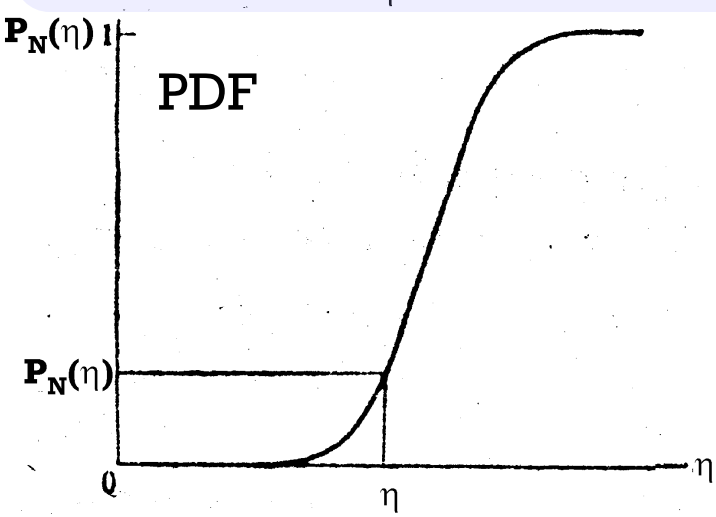
where Pr denotes probability. So the PDF is ***the probability of a random variable N takes on a value less than to equal to η.***

$$P_N(\eta_1) \leq P_N(\eta_2), \text{ for } \eta_1 \leq \eta_2.$$

$$0 \leq P_N(\eta) \leq 1, P_N(-\infty) = 0, P_N(\infty) = 1$$

Probability Density Function of Random Variables





- For a **continuous** random variable, the distribution of the variable can be uniquely specified with the **probability density function** is defined (pdf) as

$$p_N(\eta) = \frac{dP_N(\eta)}{d\eta} .$$

- It is the probability of the random variable take on a value within a unit space around a given value η.

$$p_N(\eta) \geq 0 ,$$

$$\int_{-\infty}^{\infty} p_N(\eta)\, d\eta = 1 ,$$

$$P_N(\eta) = \int_{-\infty}^{\eta} p_N(u)\, du .$$

Random Variables

- Two most commonly used parameters for describing an random variable

- ***Expected value*** (or the ***Mean*** value):

$$\mu_N = \mathrm{E}[N] = \int_{-\infty}^{\infty} \eta p_N(\eta)\, d\eta\,,$$

it can be viewed as the ***weighted average*** of the random variable. It is normally used to describe the "***center***" of the distribution taken by this variable.

- ***Variance***:

$$\sigma_N^2 = \mathrm{Var}[N] = \mathrm{E}[(N-\mu_N)^2] = \int_{-\infty}^{\infty} (\eta-\mu_N)^2 p_N(\eta)\, d\eta\,,$$

it provide a measure of how much deviation one would expect between the actual value of the random variable and the expected (mean) value.

Covered in lecture

Covariance of Random Variables

Covariance provides a measure of ***the strength of the correlation*** between two or more random variables. The covariance for two random variables and, each with sample size, is defined by the expectation value

Covariance of two random variables N_1, N_2 is defined as:

$$Cov(N_1, N_2) = E\left[\left(N_1 - \bar{N}_1\right)\left(N_2 - \bar{N}_2\right)\right]$$

where E[.] is the expectation

Covered in lecture

Discrete Random Variable

- For a random variable that only takes on a discrete set of values, its distribution can be specified using the probability mass function (PMF)

$$\Pr[N = \eta_i], \text{ for } i = 1, 2, \ldots, k,$$

where $\Pr[N = \eta_i]$ is the probability that random variable N will take on the particular value η_i.

- The probability mass function (PMF) has the following properties

$$0 \leq \Pr[N = \eta_i] \leq 1, \quad \text{for } i = 1, 2, \ldots, k,$$

$$\sum_{i=1}^{k} \Pr[N = \eta_i] = 1,$$

$$P_N(\eta) = \Pr[N \leq \eta] = \sum_{\text{all } \eta_i \leq \eta} \Pr[N = \eta_i].$$

Independent Random Variable

- For a collection of multiple random variables $N_1, N_2, \ldots N_m$,

  Having the pdf' s $p_1(\eta), p_2(\eta), \ldots, p_m(\eta)$,

- The mean and variance of the sum of these variables are

$$\mu_S = \mu_1 + \mu_2 + \cdots + \mu_m \,,$$

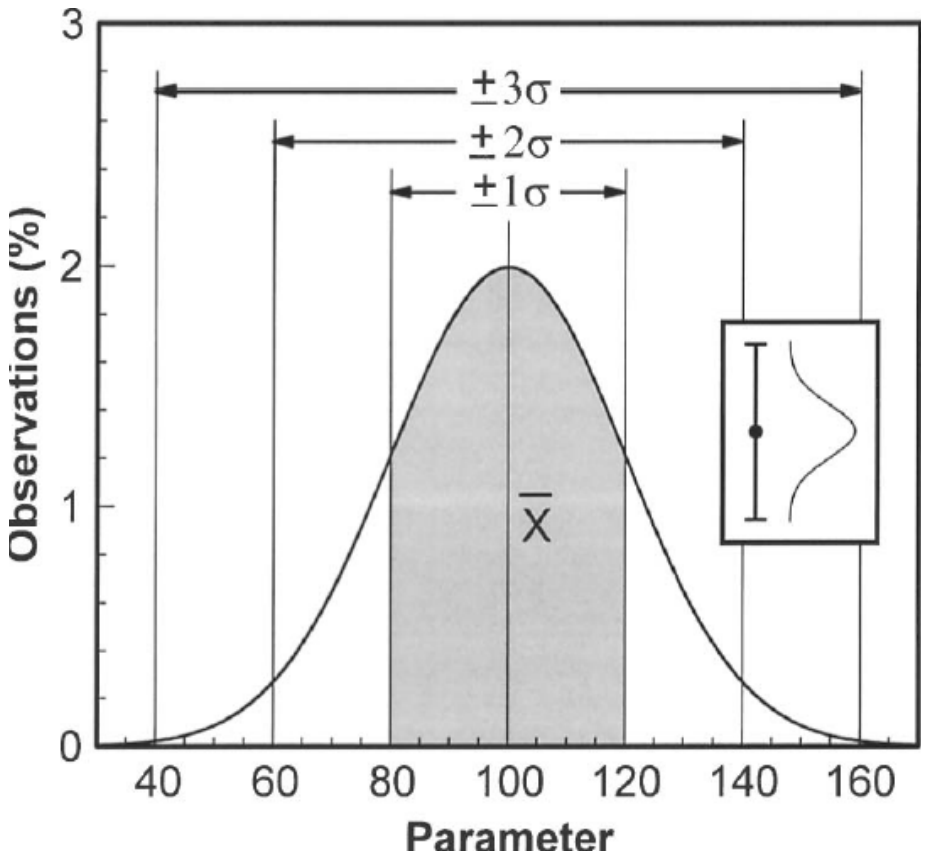
$$\sigma_S^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_m^2 \,.$$

- And similarly

$$p_S(\eta) = p_1(\eta) * p_2(\eta) * \cdots * p_m(\eta) \,,$$

Gaussian Random Variable

- In general, the mean and variance do not fully specify a random variable. However, in the case of a Gaussian variable, its pdf is uniquely determined given the mean and variance.



An Typical Emission Tomography System Described in Matrix Form

The response function of the imaging system for an impulse signal at a given source location – The Impulse Response Function h_i

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_M \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{1N} \\ p_{21} & p_{22} & p_{23} & \cdots & p_{2N} \\ p_{31} & p_{32} & p_{33} & \cdots & p_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{M1} & p_{M2} & p_{M3} & \cdots & p_{MN} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_N \end{pmatrix}$$

No. of counts observed on a given detector pixel

p_{mn}: the probability of a gamma ray generated at source pixle m being detected by detector pixel n.

f_n: No. of gamma rays generated in a given source pixel

Inverse Radon Transform

An estimate of the original image f(x,y) can be obtained as

$$\hat{f}(r,\theta) = \int_0^{\pi}\int_{-\infty}^{\infty} |\omega| P_\phi(\omega) \exp[i\omega(x\cos\phi + y\sin\phi)]\, d\omega\, d\phi$$

$$= \int_0^{\pi} p_\phi^*(x')\, d\phi$$

where

$$p_\phi^*(x') = \int_{-\infty}^{\infty} |\omega| P_\phi(\omega) \exp(i\omega x')\, d\omega$$

$$= \mathscr{F}_1^{-1}\left[|\omega| P_\phi(\omega)\right]$$

$$= \mathscr{F}_1^{-1}[|\omega|] * p_\phi(x')$$

and

The Central Slice Theorem

$$P_\phi(\omega) = F(\omega\cos\phi, \omega\sin\phi)$$

$$= F(\omega_{x'}, \omega_{y'})\big|_\phi \quad \text{or} \quad F(\omega_x, \omega_y)\big|_\phi$$

$$= F(\omega, \phi)$$

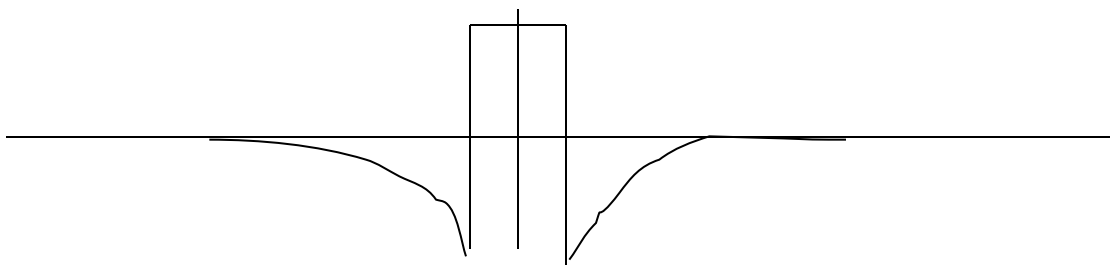
Filtered Back-projection

The Ram-Lak filter in spatial domain

$$h_{\mathrm{RL}}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{\mathrm{RL}}(\omega)\exp(ix\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B} |\omega|\,\exp(ix\omega)\,d\omega$$

$$= 2B^2\,\mathrm{sinc}(2\pi Bx) - B^2\,\mathrm{sinc}^2(\pi Bx)$$



$$\hat{f}(x,y) = \frac{1}{\pi}\int_0^{\pi} d\phi \int_{-\infty}^{\infty} dx'\, p_\phi(x')h(x\cos\phi + y\sin\phi - x')$$

Filtered Back-projection

The Ram-Lak filter in spatial domain

$$h_{\mathrm{RL}}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_{\mathrm{RL}}(\omega)\exp(ix\omega)\,d\omega$$

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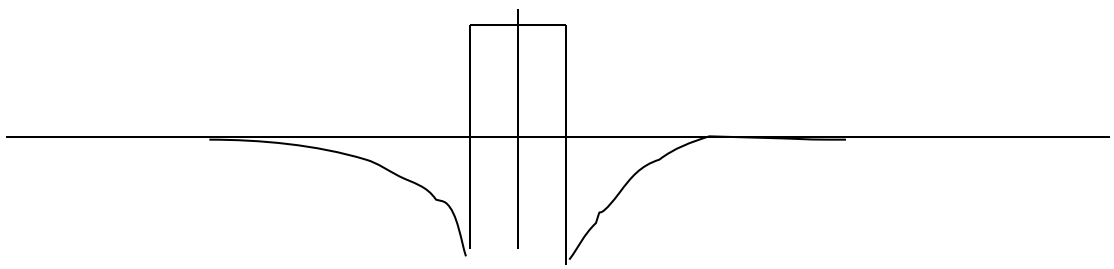

$$\hat{f}(x,y) = \frac{1}{\pi}\int_0^{\pi} d\phi \int_{-\infty}^{\infty} dx'\, p_\phi(x')h(x\cos\phi + y\sin\phi - x')$$

Filtered Back-projection

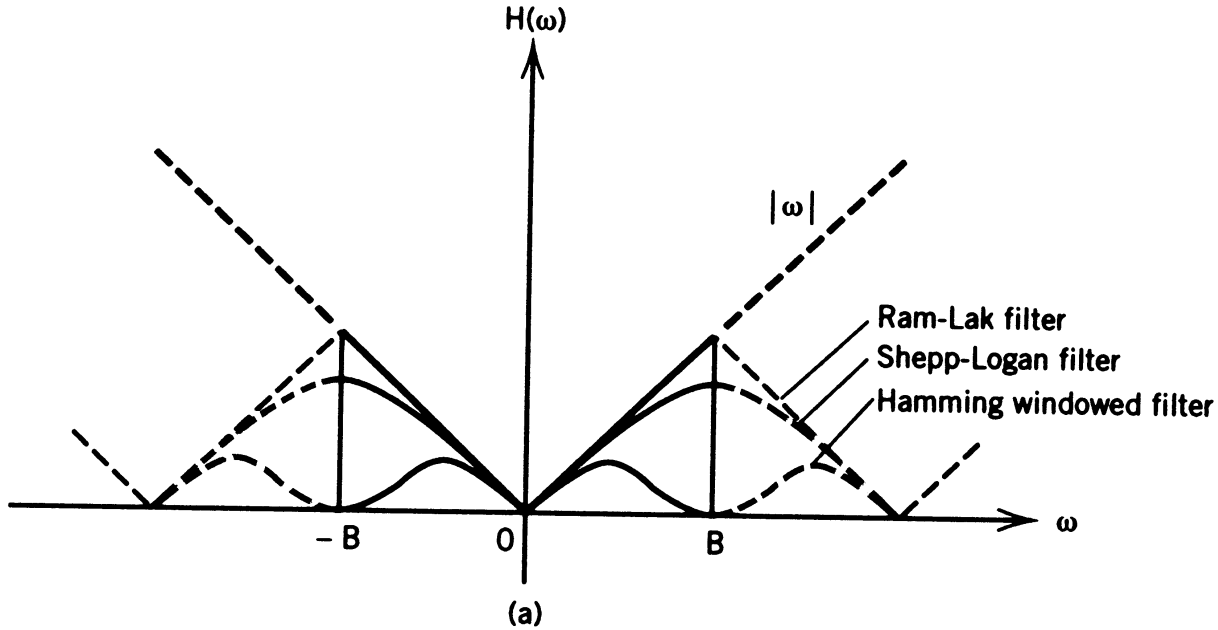

Figure 3-4 (a) Examples of the band-limited filter function of sampled data. Note the cyclic repetitiveness of the digital filter.

Why Gaussian Random Variable is Important?

- When a quantity is derived as ***the result of a large number of accumulative effects***, and each effect has a small contribution to the final outcome, then the distribution of the quantity tends to follow Gaussian distribution.
- The measured value on a detector is the result of accumulated interactions during an measurements session…
- The reconstructed image at a given pixel is the sum of the contributions from the data acquired with a large number of detector elements in the system…

$$\hat{f}_i = \sum_{j=1}^{M} q_{ij} g_j$$

- It should follow Gaussian distribution

In this simple case, we know exactly what is the probability of a given reconstructed image!!

Mean: $\bar{\hat{\mathbf{f}}} = \mathbf{H}^* \bar{\mathbf{g}}$

Covariance: $Cov(\hat{\mathbf{f}}) = \mathbf{H}^* Cov(\mathbf{p}) \left(\mathbf{H}^*\right)^T$

$Cov(.)$ is the covariance matrix of the given random vector

Covered in lecture

Signal-to-Noise Ratio (SNR)

- The ***Amplitude SNR***

$$\mathrm{SNR_a} = \frac{\mathrm{Amplitude}(f)}{\mathrm{Amplitude}(N)}.$$

- There are many ways to define the amplitude SNR.
- The amplitude of the noise is normally refer to as the standard deviation of the random fluctuation associated with the signal.

- The ***Power SNR***

$$\mathrm{SNR_p} = \frac{\mathrm{power}(f)}{\mathrm{power}(N)}.$$

$$power\left[f(x,y)\right] = \left|F(u\right.$$

$$= F^2(u,v) + F^2(u,v)$$

- Again, the exact definition of the power SNR also depends on the definition of the signal power and noise power.

Covered in lecture

Magnitude and Phase

- In general, Fourier transform is a complex valued signal, even if f(x,y) is real valued.
- It is sometimes useful to consider the *magnitude* and *phase* of the Fourier transform separately.

Fourier coefficients are complex: $$F(u,v) = F_R(u,v) + j \cdot F_I(u,v)$$

Magnitude: $$|F(u,v)| = \sqrt{F_R^2(u,v) + F_I^2(u,v)}$$

Phase: $$\angle F(u,v) = \tan^{-1} \frac{F_I(u,v)}{F_R(u,v)}$$

An alternative representation: $$F(u,v) = |F(u,v)| e^{j\angle F(u,v)}$$

- The square of the magnitude $|F(u,v)|^2$ is referred to as the *power spectrum* of the original function.

Covered in lecture