Fourier Transform and Sampling

**Reading Material:**
Fourier Transform Basics
A Fourier Transform is an integral transform that re-expresses a function in terms of different sine waves of varying amplitudes, wavelengths, and phases.

So what does this mean exactly?

Let’s start with an example…in 1-D

Can be represented by:

When you let these three waves interfere with each other you get your original wave function!

Since this object can be made up of 3 fundamental frequencies an ideal Fourier Transform would look something like this:

Notice that it is symmetric around the central point and that the amount of points radiating outward correspond to the distinct frequencies used in creating the image.

Covered in lecture
Fourier Transform – What and Why?

What is Fourier Transform?

• A function can be described by a summation of waves with different frequency, amplitudes and phases.

The importance of Fourier Transform in Imaging?

• Signal representations in the frequency domain provide unique information.

• Certain computations can be performed more efficiently in frequency domain.

• Certain hardware naturally measures signals in the frequency domain.
Continuous Fourier Transform

- For any square-integrable function \( f(x,y) \), a **continuous Fourier transform** is defined as

\[
F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)e^{-j2\pi(ux+vy)} \, dx \, dy
\]

where \( j = \sqrt{-1} \)

- We can also define an **inverse Fourier transform** as

\[
f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v)e^{j2\pi(ux+vy)} \, du \, dv
\]

- Both \( f(x,y) \) and \( F(u,v) \) have infinite support.
- Both \( f(x,y) \) and \( F(u,v) \) are defined on a continuum of values.
- \( f(x,y) \) and \( F(u,v) \) must contain the same information.
Fourier Transform and Spatial Frequency

- **Fourier transform** can be viewed as a decomposition of the function \( f(x, y) \) into a linear combination of complex exponentials with strength \( F(u, v) \).

\[
f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v)e^{j2\pi(ux+vy)} \, dudv
\]

- Fourier transform provides information on the sinusoidal composition of a signal at different spatial frequencies.

\[
e^{j2\pi(ux+vy)} = \cos[2\pi(ux + vy)] + j \sin[2\pi(ux + vy)]
\]

- \( F(u,v) \) is normally referred to as the spectrum of the function \( f(x,y) \).
Fourier transform provides information on the sinusoidal composition of a signal at different spatial frequencies.
What is Spatial Frequency?

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v)e^{-j2\pi(ux+vy)} \, dudv \]

\[ e^{-j2\pi(ux+vy)} \]

\[ = \cos[2\pi(ux + vy)] + j \sin[2\pi(ux + vy)] \]
Point Impulse Signal

- A point source is mathematically represented by the delta function or Dirac function.

\[
\delta(x, y) \begin{cases} 
\neq 0, & x = 0 \text{ and } y = 0 \\
0, & \text{otherwise} 
\end{cases}
\]

and

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x, y) \, dx \, dy = 1
\]

- The sampling property

\[
f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \cdot \delta(x - \xi, y - \eta) \, d\xi \, d\eta
\]
What is Spatial Frequency?

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v)e^{-j2\pi(ux+vy)} \, dudv \]

\[ e^{-j2\pi(ux+vy)} \]

\[ = \cos[2\pi(ux + vy)] + j \sin[2\pi(ux + vy)] \]
An Example
Examples

Delta function

2-D DC plane

2-D line impulse

2-D line impulse

Square signal

2-D sinc function

Covered in lecture
## Basic Fourier Transform Pairs

<table>
<thead>
<tr>
<th>Signal</th>
<th>Fourier Transform</th>
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Properties of Fourier Transform (1)

- **Linearity** \[ F[a_1 f(x, y) + a_2 g(x, y)] = a_1 F[f(x, y)] + a_2 F[g(x, y)] \]

\[ F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux + vy)} \, dx \, dy \]

where \( j = \sqrt{-1} \)

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux + vy)} \, du \, dv \]
Magnitude and Phase

- In general, Fourier transform is a complex valued signal, even if \( f(x,y) \) is real valued.
- It is sometimes useful to consider the **magnitude** and **phase** of the Fourier transform separately.

**Fourier coefficients are complex:**

\[
F(u, v) = F_R(u, v) + j \cdot F_I(u, v)
\]

**Magnitude:**

\[
|F(u, v)| = \sqrt{F_R^2(u, v) + F_I^2(u, v)}
\]

**Phase:**

\[
\angle F(u, v) = \tan^{-1} \frac{F_I(u, v)}{F_R(u, v)}
\]

**An alternative representation:**

\[
F(u, v) = |F(u, v)| e^{j \angle F(u, v)}
\]

- The square of the magnitude \(|F(u,v)|^2\) is referred to as the **power spectrum** of the original function.
Properties of Fourier Transform (2)

**Shifting Property** – Shift in spatial domain is equivalent to phase change in spatial frequency domain.

\[
\mathcal{F}\left[ f(x - x_0, y - y_0) \right] = \mathcal{F}\left[ f(u, v) \right] e^{-j2\pi(ux_0 + vy_0)}
\]

An example

\[
F(u,v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} \, dx \, dy
\]

where \( j = \sqrt{-1} \)

\[
\Lambda(x/16)\Lambda(y/16)
\]

real and even
Real\{F(u,v)\} = 256 \text{sinc}^2(16u)\text{sinc}^2(16v)  \quad \text{Imag}\{F(u,v)\} = 0

\left| F(u,v) \right| = \sqrt{\left\{\text{Real}\left[ F(u,v) \right] \right\}^2 + \left\{\text{Imag}\left[ F(u,v) \right] \right\}^2}

= 256 \text{sinc}^2(16u)\text{sinc}^2(16v)

\angle F(u,v) = \tan^{-1} \frac{\text{Imag}[F(u,v)]}{\text{Real}[F(u,v)]} = 0
Properties of Fourier Transform

\[ |F(u,v)| = 256 \text{sinc}^2(16u)\text{sinc}^2(16v) \]

and

\[ \angle F(u,v) = -2\pi \]

\[ \Lambda[(x-1)/16] \Lambda[y/16] \]
shifted by 1
Properties of Fourier Transform

1-D Gaussian function: \( f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \left( \frac{x}{\sigma} \right)^2} \)

**Spatial Domain**

A Gaussian transforms to a Gaussian

**Spatial Frequency Domain**

\( \mathcal{F} [f(x - x_0, y - y_0)] = \mathcal{F} [f(u, v)] e^{-j2\pi(ux_0 + vy_0)} \)

Spectral phase is zero

Magnitude is a Gaussian

Covered in lecture
Properties of Fourier Transform

\[ F[f(x - a)] = F[f(x)] \exp(-j2\pi u a) \]

**Spatial Domain**

\[
F(u, v) = F_R(u, v) + j \cdot F_I(u, v)
\]

\[
|F(u, v)| = \sqrt{F_R^2(u, v) + F_I^2(u, v)}
\]

\[
\angle F(u, v) = \tan^{-1} \frac{F_I(u, v)}{F_R(u, v)}
\]

\[
F(u, v) = |F(u, v)| e^{j\angle F(u, v)}
\]

**Spatial Frequency Domain**

Linear shifting in spatial domain simply adds some linear phase to the pulse

Magnitude is unchanged
Properties of Fourier Transform

- **Scaling**

\[ F[f(ax, by)] = \frac{1}{ab} F\left(\frac{u}{a}, \frac{v}{b}\right) \]

**An 1-D example**

The shorter the pulse, the broader the spectrum!

\[ F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi(ux+vy)} \, dx \, dy \]

where \( j = \sqrt{-1} \)

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) e^{j2\pi(ux+vy)} \, du \, dv \]

Spatial domain, \( f(x) \)  
Fourier Transform, \( F(u) \)
Properties of Fourier Transform

- **Scaling**

\[ F[f(ax, by)] = \frac{1}{ab} F\left(\frac{u}{a}, \frac{v}{b}\right) \]

An 2-D example
Linear and Shift Invariant Systems Revisited

For a shift-invariant system, the output is the input convolved with the impulse response function.

\[
g(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) h(x, y, \xi, \eta) d\xi d\eta
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) h(x - \xi, y - \eta) d\xi d\eta
\]

\[
= f(x, y) * h(x, y)
\]

where \( h(x, y, \xi, \eta) \) is the response of the system to an delta impulse signal.
Convolution Theorem

• The convolution of two 2-D functions is defined as

\[ f(x, y) \ast h(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \cdot h(x - \xi, y - \eta) d\xi d\eta \]

• The Fourier transform of the convolution is equal to the product of the individual Fourier transforms:

\[ \mathcal{F}[f(x, y) \ast h(x, y)] = \mathcal{F}[f(x, y)] \cdot \mathcal{F}[h(x, y)] \]

where \( \mathcal{F}[\cdot] \) is the Fourier transform operator.

The output from a shift invariant system is therefore

\[ g(x, y) = f(x, y) \ast h(x, y) = \mathcal{F}^{-1} \{ \mathcal{F}[f(x, y)] \cdot \mathcal{F}[h(x, y)] \} \]

Covered in lecture
Properties of Fourier Transform

- **Product**

\[
F[f(x, y) \cdot g(x, y)] = F[f(x, y)] \ast F[g(x, y)]
\]

Fourier transform of the product of two functions equals to the convolution of the Fourier transforms of individual functions.

- **The Convolution Theorem**

\[
F[f(x, y) \ast g(x, y)] = F[f(x, y)] \cdot F[g(x, y)]
\]
Convolution Theorem

Proof:

\[
\mathbf{F}[f(x, y) * g(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \cdot g(x - \xi, y - \eta) \, d\xi \, d\eta \right) e^{-j2\pi(ux + vy)} \, dx \, dy
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \cdot g(x', y') \cdot e^{-j2\pi[u(x' + \xi) + v(y' + \eta)]} \, dx' \, dy' \right) \, d\xi \, d\eta
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \cdot e^{-j2\pi(u\xi + v\eta)} \cdot g(x', y') \cdot e^{-j2\pi(u'x' + v'y')} \, dx' \, dy' \right) \, d\xi \, d\eta
\]

\[
= \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \cdot e^{-j2\pi(u\xi + v\eta)} \, d\xi \, d\eta \right] \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x', y') \cdot e^{-j2\pi(u'x' + v'y')} \, dx' \, dy' \right]
\]

\[
= \mathbf{F}[f(x, y)] \cdot \mathbf{F}[g(x, y)]
\]
\[
F[f(x, y) \cdot g(x, y)] = F[f(x, y)] \ast F[g(x, y)]
\]

**Proof:**

The convolution theorem states:

\[
F[f(x, y) \ast g(x, y)] = F[f(x, y)] \cdot F[g(x, y)]
\]

Similarly we can prove that:

\[
F^{-1}[f(x, y) \ast g(x, y)] = F^{-1}[f(x, y)] \cdot F^{-1}[g(x, y)]
\]

If we define:

\[
F = F^{-1}[f(x, y)], G = F^{-1}[g(x, y)], \text{ then } f(x, y) = F[F], g(x, y) = F[G]
\]

We can see that

\[
F^{-1}\{F[F] \ast F[G]\} = F \ast G
\]

Therefore

\[
F[F \ast G] = F[F] \ast F[G]
\]

Note that \( F \) and \( G \) are arbitrary function, so that we can re-write the above equation as

\[
F[f(x, y) \cdot g(x, y)] = F[f(x, y)] \ast F[g(x, y)]
\]
Properties of Fourier Transform

- **Separable product**

  \[
  f(x, y) = f_1(x) \cdot f_2(y)
  \]

  then

  \[
  \mathcal{F}_{u,v}[f(x, y)] = \mathcal{F}_u[f_1(x)] \cdot \mathcal{F}_v[f_2(y)]
  \]

  \[
  \mathcal{F}\{f(x, y)\} = \text{sinc}(u)\text{sinc}(v)
  \]

  \[
  f(x, y) = \text{Rect}(x)\text{Rect}(y)
  \]
Symmetry Properties

Expanding the Fourier transform of a function, \( f(t) \):

\[
F(\omega) = \int_{-\infty}^{\infty} [\text{Re}\{f(t)\} + i \text{Im}\{f(t)\}] [\cos(\omega t) - i \sin(\omega t)] \, dt
\]

Expanding more, noting that:

- \( \int O(t) \, dt = 0 \) if \( O(t) \) is an odd function
- \( \int O(t) \, dt = 0 \) if \( O(t) \) is an even function

\[
F(\omega) = \int_{-\infty}^{\infty} \text{Re}\{f(t)\} \cos(\omega t) \, dt + \int_{-\infty}^{\infty} \text{Im}\{f(t)\} \sin(\omega t) \, dt \leftarrow \text{Re}\{F(\omega)\}
\]

\[
= 0 \text{ if } \text{Re}\{f(t)\} \text{ is odd} \quad = 0 \text{ if } \text{Im}\{f(t)\} \text{ is even}
\]

\[
+ i \int_{-\infty}^{\infty} \text{Im}\{f(t)\} \cos(\omega t) \, dt - i \int_{-\infty}^{\infty} \text{Re}\{f(t)\} \sin(\omega t) \, dt \leftarrow \text{Im}\{F(\omega)\}
\]

Even functions of \( \omega \)  
Odd functions of \( \omega \)

Covered in lecture
Fourier transform of Circularly Symmetric Functions (1)

A function is **circularly symmetric** if

\[ f_{\theta}(x, y) = f(x, y), \quad \text{for every } \theta, \]

where \( f_{\theta}(x, y) \) is a rotated version of \( f(x, y) \).

In this case, the function

\[ f(x, y) = f(r), \quad \text{where } r = \sqrt{x^2 + y^2} \]

The Fourier transform is also **real** and **circularly symmetric**.

\[ |F(u, v)| = F(u, v) \text{ and } \angle F(u, v) = 0 \]

and

\[ F(u, v) = F(q), \quad \text{where } q = \sqrt{u^2 + v^2} \]
Fourier Transform of Circularly Symmetric Functions (2)

The Fourier transform of a *circularly symmetric* function is called Hankel transform

\[
F(q) = 2\pi \int_{-\infty}^{\infty} f(r) J_0(2\pi qr) rdr
\]

where \( J_n(r) \) is the zero-order Bessel function,

\[
J_n(r) = \frac{1}{\pi} \int_{0}^{\pi} \cos(nr - r \sin \phi) d\phi, \quad n = 0, 1, 2, \ldots,
\]

The inverse Hankel transform is given by

\[
f(r) = 2\pi \int_{-\infty}^{\infty} F(q) J_0(2\pi qr) q dq
\]
Properties of Fourier Transform (4)

- **Parseval’s Theorem**

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(x, y)|^2 \, dx \, dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |F(u, v)|^2 \, du \, dv
\]

Total energy of a signal of a signal \( f(x, y) \) in spatial domain equals its total energy in spatial frequency domain.

Fourier transform and its inverse is energy preserving!
Convolution Theorem Revisited

The convolution theorem enables one to perform convolution operation as multiplication process in spatial frequency domain.

\[ f(x, y) * h(x, y) = F^{-1} \left\{ F[f(x, y)] \cdot F[h(x, y)] \right\} \]

By using the **Fast Fourier Transform** (FFT) algorithms, the convolution operation can be performed very efficiently!! This provide a practical way for modeling linear shift-invariant systems …

\[ g(x, y) = f(x, y) * h(x, y) \]
System Transfer Function

The Fourier transform of the impulse response function $h$ is called the \textit{system transfer function}.

$$H(u, v) = F[h(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y)e^{-2\pi j(ux + vy)} \, dx \, dy$$

$$G(u, v) = F(u, v)H(u, v)$$
System Transfer Function

\[ G(u, v) = F(u, v)H(u, v) \]

An *ideal low-pass filter* is defined as

\[
H(u, v) = \begin{cases} 
1 & \text{for } \sqrt{u^2 + v^2} \leq c \\
0 & \text{for } \sqrt{u^2 + v^2} > c
\end{cases}
\]

\(c\) is called the *cut-off frequency*. 
System Transfer Function

Figure 2.12
The response of an ideal low-pass filter for two values of the cutoff frequency $c$ ($c_1 > c_2$).
Spatial Frequency Revisited

Image

Fourier Space (log magnitude)

Detail

Contrast

Covered in lecture
Relation Between 1-D and 2-D Fourier Transforms

\[ F(u, v) = \int_{-\infty}^{\infty} e^{-i \cdot 2\pi \cdot vy} \left[ \int_{-\infty}^{\infty} f(x, y) \cdot e^{-i \cdot 2\pi \cdot ux} \, dx \right] \, dy \]

Rearranging the Fourier Integral,

Taking the integrals along \( x \) gives, \( \hat{F}(u, y) \)

Taking the integrals of \( \hat{F}(u, y) \) along \( y \) gives \( F(u, v) \)

\[ F(u) \]

\[ F(u, v) = \int_{-\infty}^{\infty} \hat{F}(u, y) e^{-i \cdot 2\pi \cdot vy} \, dy \]
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