Fourier Transform and Sampling

Reading Material:

Chapter 2, Medical Imaging Signals and Systems, 2'nd Edition,

by Prince and Links, Prentice Hall, 2006.

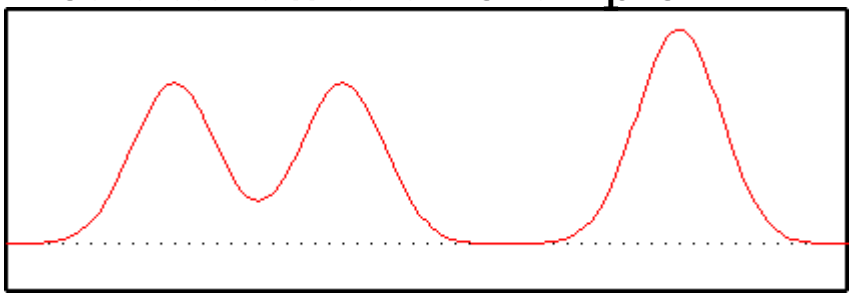
Fourier Transform Basics

Continuous Fourier Transform

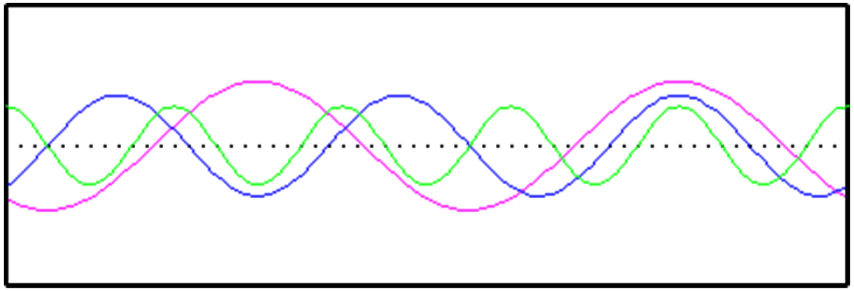
A Fourier Transform is an integral transform that re-expresses a function in terms of different sine waves of varying amplitudes, wavelengths, and phases.

So what does this mean exactly?

Let's start with an example…in 1-D

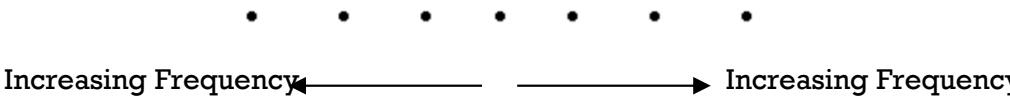


Can be represented by:



When you let these three waves interfere with each other you get your original wave function!

Since this object can be made up of 3 fundamental frequencies an ideal Fourier Transform would look something like this:



Notice that it is symmetric around the central point and that the amount of points radiating outward correspond to the distinct frequencies used in creating the image.

Covered in lecture

Fourier Transform – What and Why?

What is Fourier Transform?

- A function can be described by a summation of waves with different frequency, amplitudes and phases.

The importance of Fourier Transform in Imaging?

- Signal representations in the frequency domain provide **unique information.**
- Certain **computations** can be performed more efficiently in frequency domain.
- Certain **hardware** naturally measures signals in the frequency domain.

Covered in lecture

Continuous Fourier Transform

- For any square-integrable function $f(x,y)$, a **continuous Fourier transform** is defined as

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$where\ \ j = \sqrt{-1}$$

- We can also define an ***inverse Fourier transform*** as

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

- Both f(x,y) and F(u,v) have infinite support.
- Both f(x,y) and F(u,v) are defined on a continuum of values.
- f(x,y) and F(u,v) must contain the same information.

Covered in lecture

Fourier Transform and Spatial Frequency

- **Fourier transform** can be viewed as a **decomposition** of the function f(x,y) into a linear combination of complex exponentials with strength F(u,v).

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} du dv$$

- Fourier transform provides information on the sinusoidal composition of a signal at different ***spatial frequencies***.

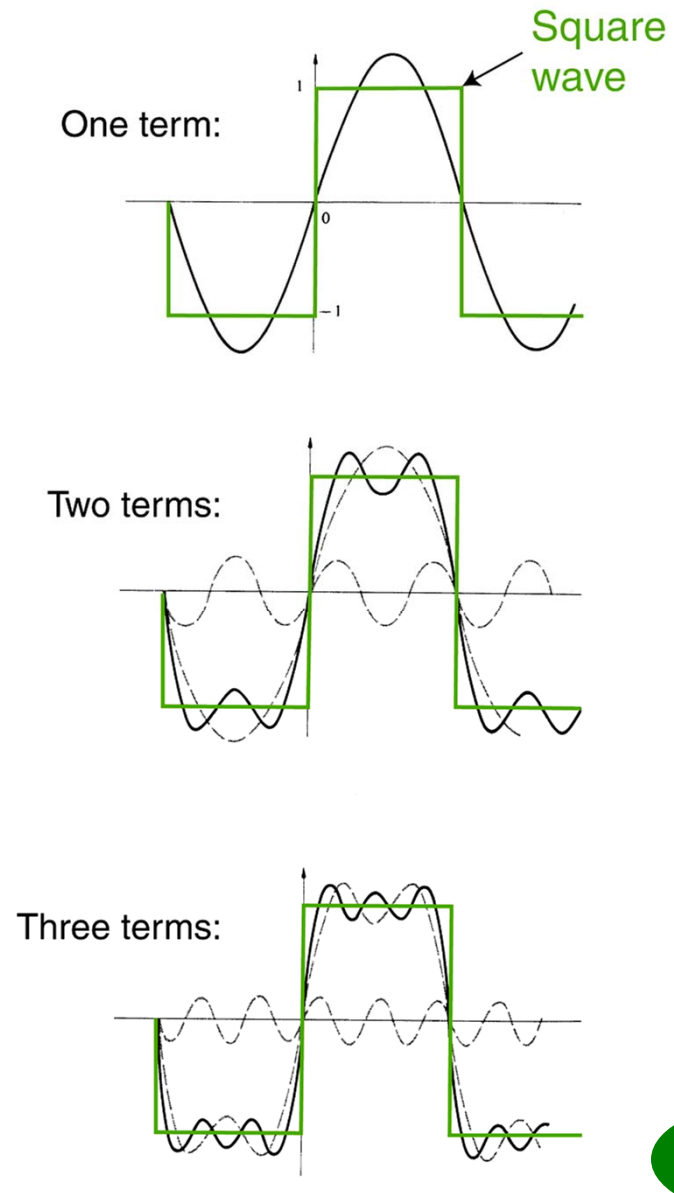
$$e^{j2\pi(ux+vy)} = \cos[2\pi(ux+vy)] + j\sin[2\pi(ux+vy)]$$

- F(u,v) is normally referred to as the ***spectrum*** of the function f(x,y).

Covered in lecture

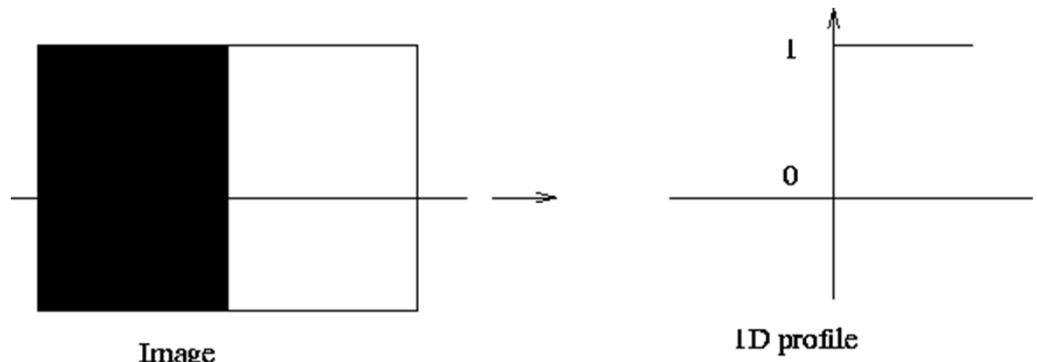
Fourier Transform and Spatial Frequency

Fourier transform provides information on the sinusoidal composition of a signal at different ***spatial frequencies***.



Covered in lecture

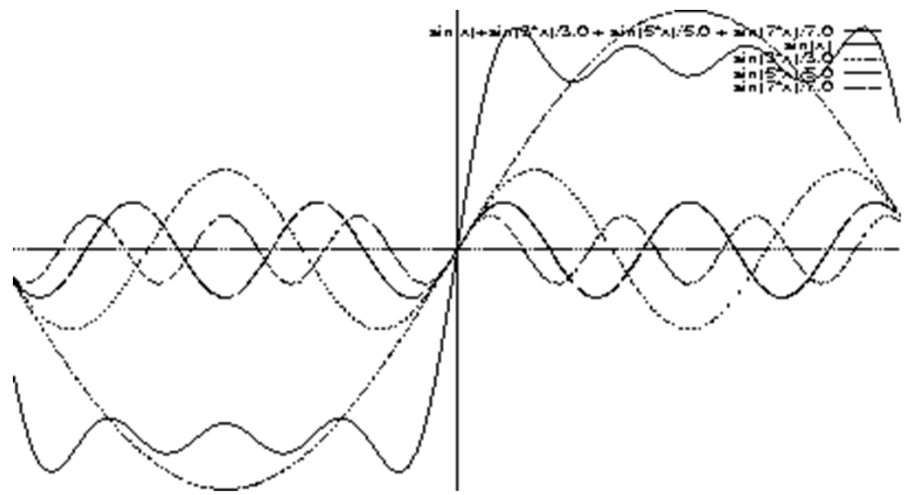
What is Spatial Frequency?



$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{-j2\pi(ux+vy)} du dv$$

$$e^{-j2\pi(ux+vy)}$$

$$= \cos[2\pi(ux+vy)] + j\sin[2\pi(ux+vy)]$$



Covered in lecture

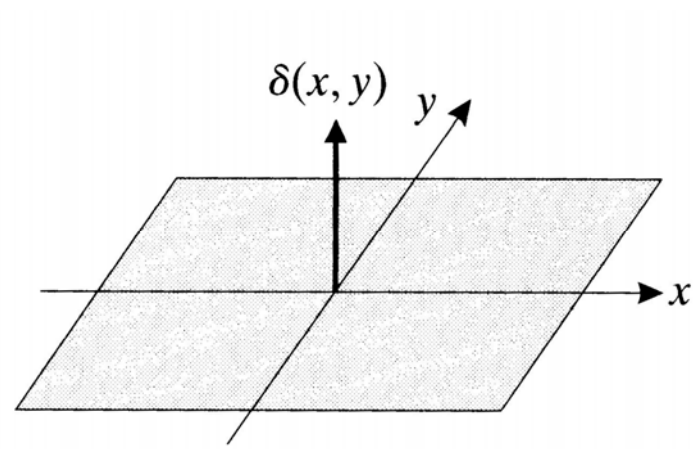
Point Impulse Signal

- A point source is mathematically represented by the **delta function** or **Dirac function**.

$$\delta(x,y)\begin{cases}\neq 0, & x=0 \text{ and } y=0\\ =0, & otherwise\end{cases}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x,y)dxdy=1$$

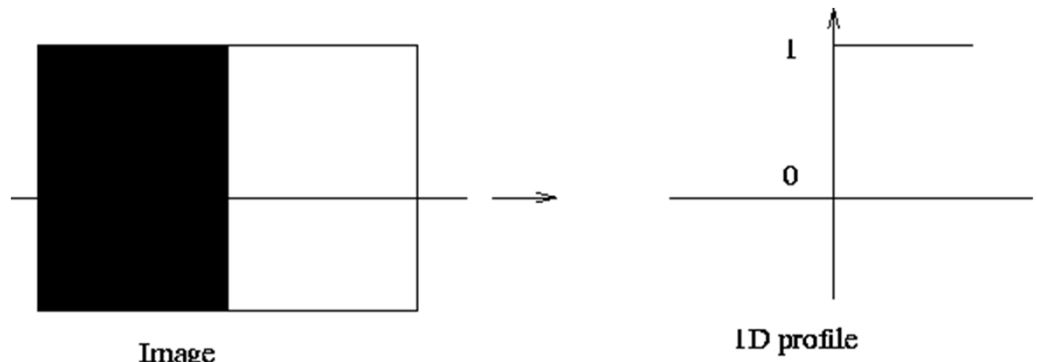


- The **sampling property**

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)\cdot\delta(x-\xi,y-\eta)d\xi d\eta$$

Covered in lecture

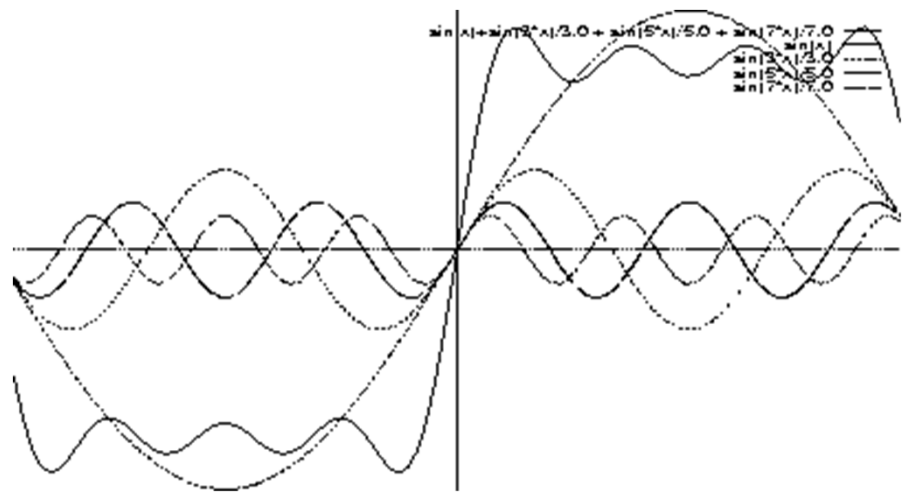
What is Spatial Frequency?



$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{-j2\pi(ux+vy)} du dv$$

$$e^{-j2\pi(ux+vy)}$$

$$= \cos[2\pi(ux+vy)] + j\sin[2\pi(ux+vy)]$$



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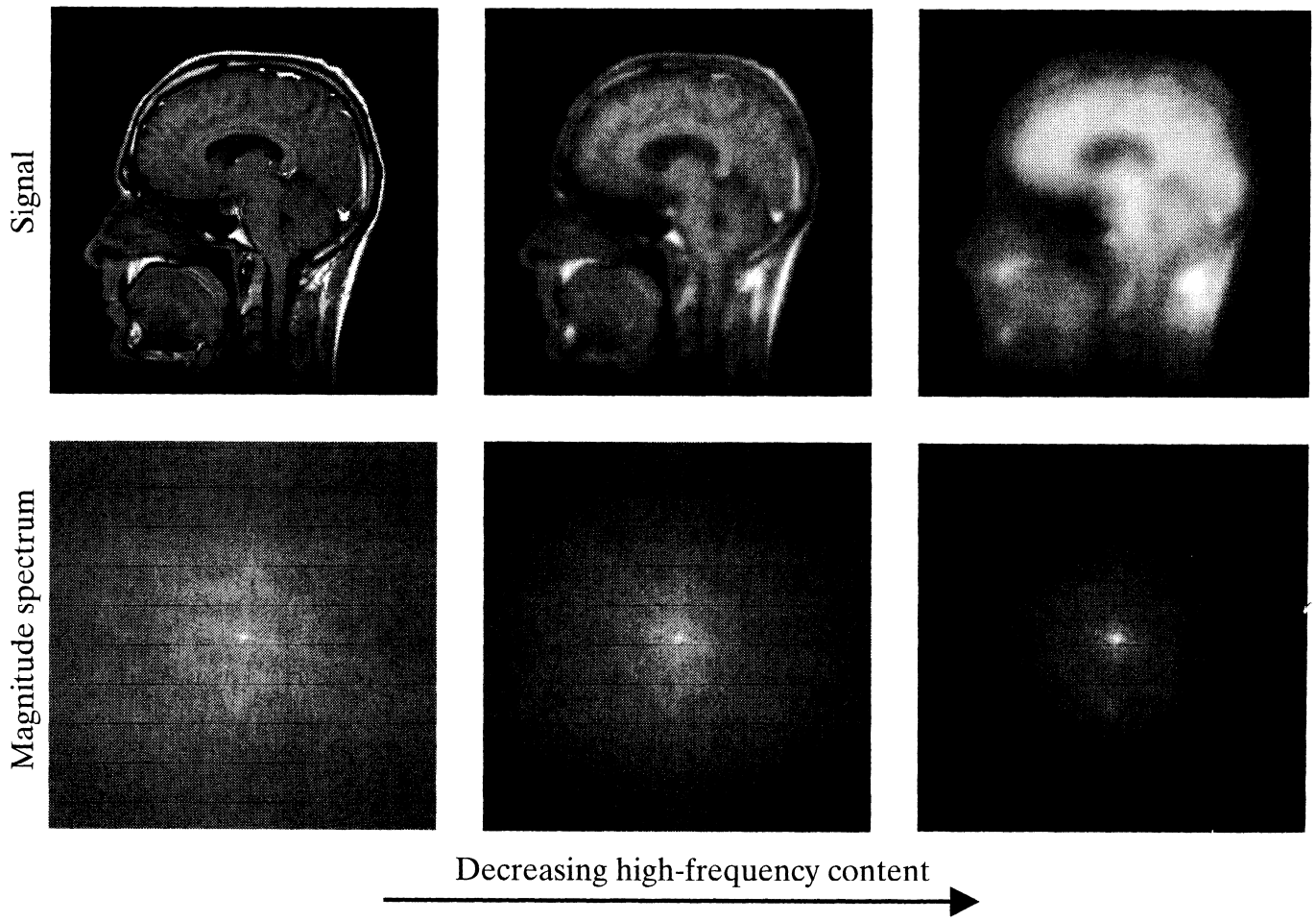
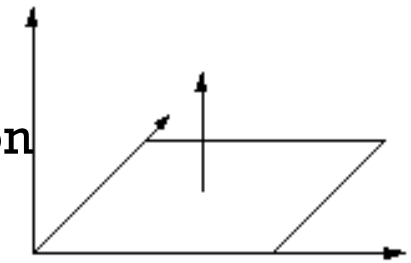
An Example



Covered in lecture

Examples

Delta function



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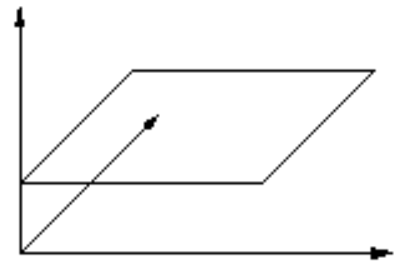
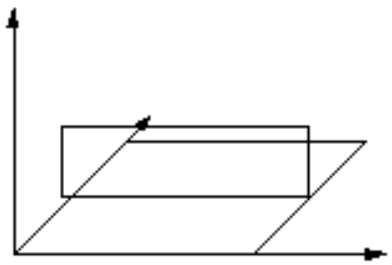


2-D DC plane

2-D line impulse



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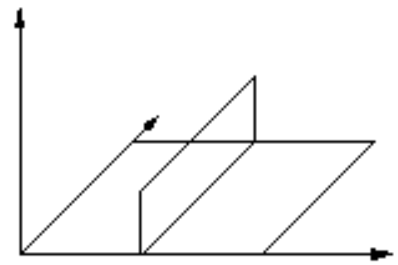
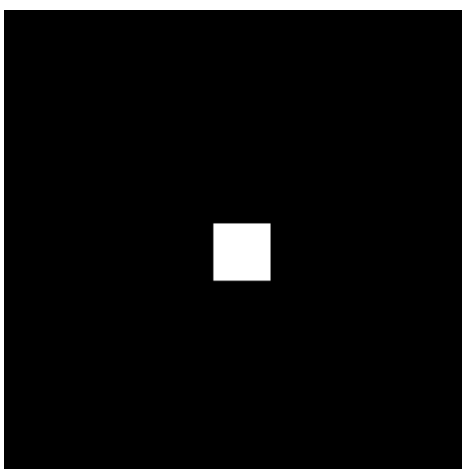
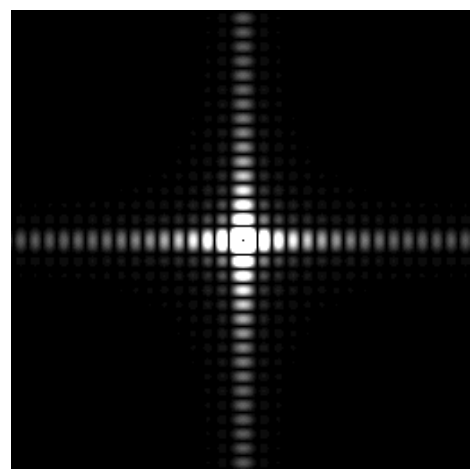


2-D line impulse

Square signal





2-D sinc function

Covered in lecture

Basic Fourier Transform Pairs

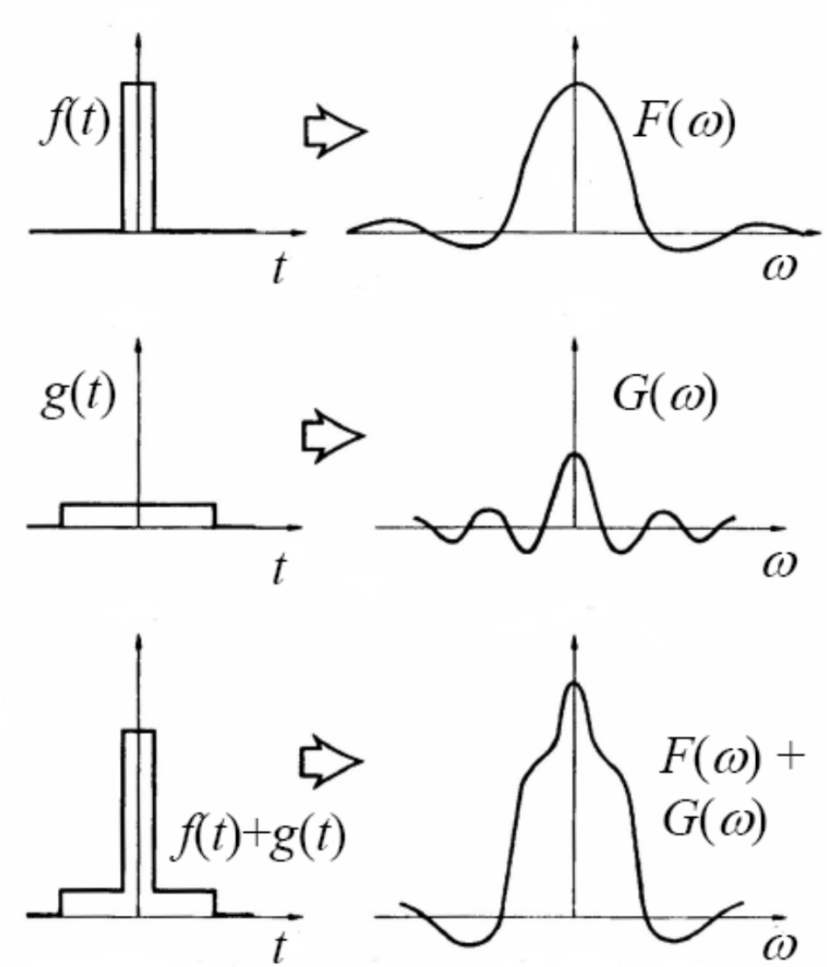
Basic Fourier transform pairs

Signal	Fourier Transform
1	$\delta(u, v)$
$\delta(x, y)$	1
$\delta(x - x_0, y - y_0)$	$e^{-j2\pi(ux_0+vy_0)}$
$\delta_s(x, y; \Delta x, \Delta y)$	$\mathrm{comb}(u\Delta x, v\Delta y)$
$e^{j2\pi(u_0x+v_0y)}$	$\delta(u - u_0, v - v_0)$
$\sin[2\pi(u_0x + v_0y)]$	$\frac{1}{2j}[\delta(u - u_0, v - v_0) - \delta(u + u_0, v + v_0)]$
$\cos[2\pi(u_0x + v_0y)]$	$\frac{1}{2}[\delta(u - u_0, v - v_0) + \delta(u + u_0, v + v_0)]$
$\mathrm{rect}(x, y)$	$\mathrm{sinc}(u, v)$
$\mathrm{sinc}(x, y)$	$\mathrm{rect}(u, v)$
$\mathrm{comb}(x, y)$	$\mathrm{comb}(u, v)$
$e^{-\pi(x^2+y^2)}$	$e^{-\pi(u^2+v^2)}$

Covered in lecture

Properties of Fourier Transform (1)

- ***Linearity*** $F\left[a_1 f(x,y) + a_2 g(x,y)\right] = a_1 F\left[f(x,y)\right] + a_2 F\left[g(x,y)\right]$



Covered in lecture

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)} dxdy$$

$$where\ j = \sqrt{-1}$$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) e^{j2\pi(ux+vy)} dudv$$

Magnitude and Phase

- In general, Fourier transform is a complex valued signal, even if f(x,y) is real valued.
- It is sometimes useful to consider the ***magnitude*** and ***phase*** of the Fourier transform separately.

Fourier coefficients are complex:

$$F(u,v) = F_R(u,v) + j \cdot F_I(u,v)$$

Magnitude:

$$|F(u,v)| = \sqrt{F_R^2(u,v) + F_I^2(u,v)}$$

Phase:

$$\angle F(u,v) = \tan^{-1} \frac{F_I(u,v)}{F_R(u,v)}$$

An alternative representation:

$$F(u,v) = |F(u,v)| e^{j\angle F(u,v)}$$

- The square of the magnitude $|F(u,v)|^2$ is referred to as the ***power spectrum*** of the original function.

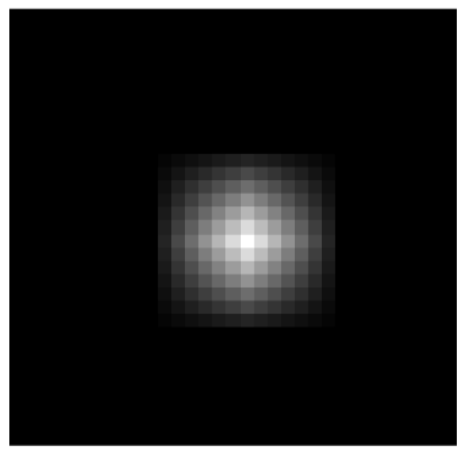
Covered in lecture

Properties of Fourier Transform (2)

Shifting Property – Shift in spatial domain is equivalent to phase change in spatial frequency domain.

$$\mathscr{F}\left[f(x-x_0, y-y_0)\right] = \mathscr{F}\left[f(u,v)\right]e^{-j2\pi(ux_0+vy_0)}$$

An example



$\Lambda(x/16)\Lambda(y/16)$

real and even

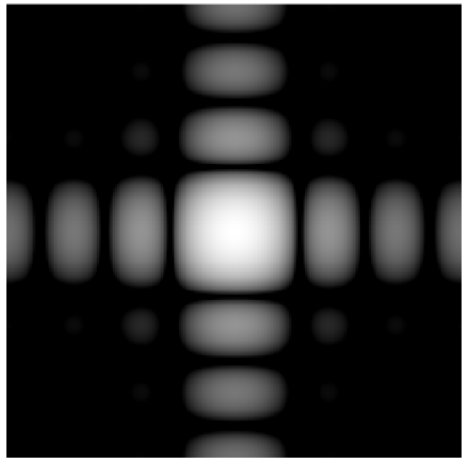
$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$where\ \ j = \sqrt{-1}$$

Covered in lecture

Properties of Fourier Transform

Shown in Log scale



$\text{Real}\{F(u,v)\} = 256\ \text{sinc}^2(16u)\text{sinc}^2(16v)$

$Imag\{F(u,v)\} = 0$

$$|F(u,v)| = \sqrt{\{Real[F(u,v)]\}^2 + \{Imag[F(u,v)]\}^2}$$

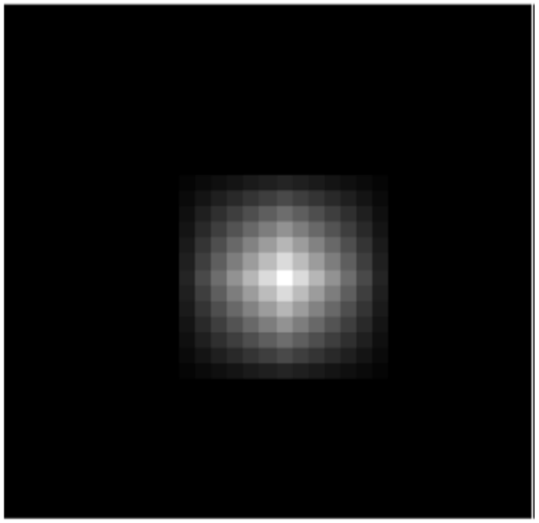
$$= 256\,\text{sinc}^2(16u)\text{sinc}^2(16v)$$

and

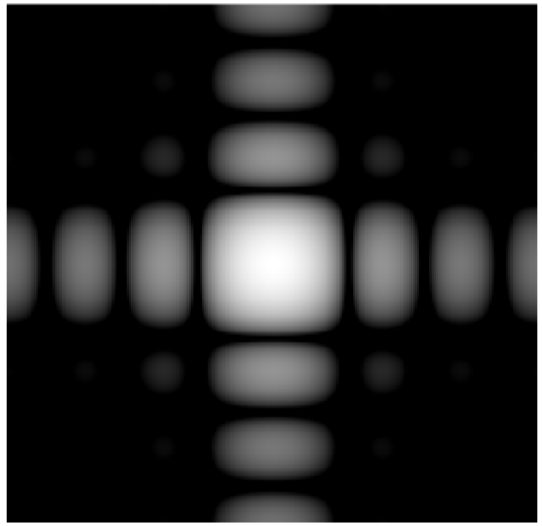
$$\angle F(u,v) = \tan^{-1}\frac{Imag[F(u,v)]}{Real[F(u,v)]} = 0$$

Covered in lecture

Properties of Fourier Transform



$\Lambda[(x-1)/16]\Lambda[y/16]$
shifted by 1



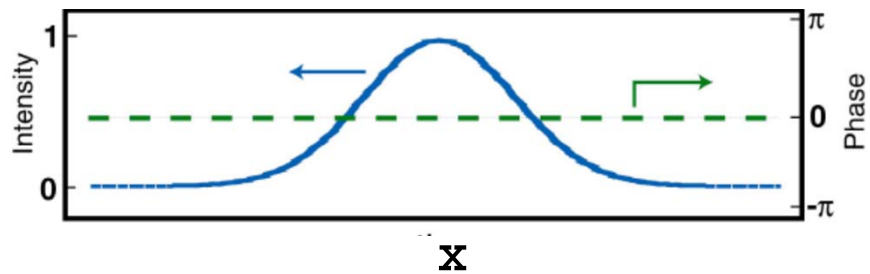
$$|F(u,v)| =$$
$$256\,\mathrm{sinc}^2(16\mathrm{u})\mathrm{sinc}^2(16\mathrm{v})$$
and
$$\angle F(u,v) = -2\pi u$$

Covered in lecture

Properties of Fourier Transform

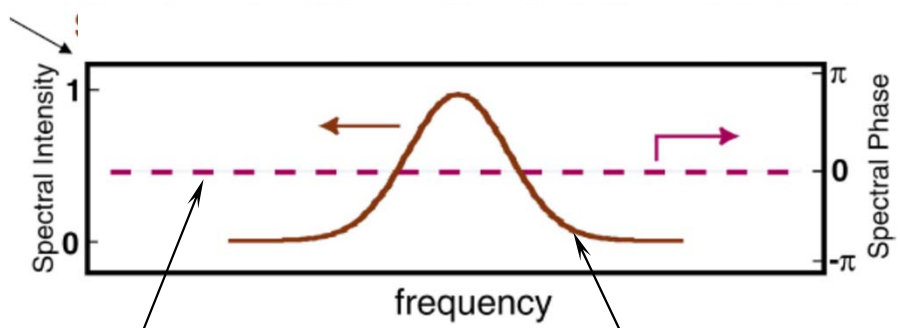
$$1\text{-D Gaussian function}: f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

Spatial Domain



A Gaussian transforms to a Gaussian

Spatial Frequency Domain



$$\mathscr{F}\left[f(x-x_0, y-y_0)\right] = \mathscr{F}\left[f(u,v)\right]e^{-j2\pi(ux_0+vy_0)}$$

Spectral phase is zero

Covered in lecture

Magnitude is a Gaussian

Properties of Fourier Transform

$$F[f(x-a)] = F[f(x)]exp(-j2\pi ua)$$

Spatial Domain

$$F(u,v) = F_R(u,v) + j \cdot F_I(u,v)$$

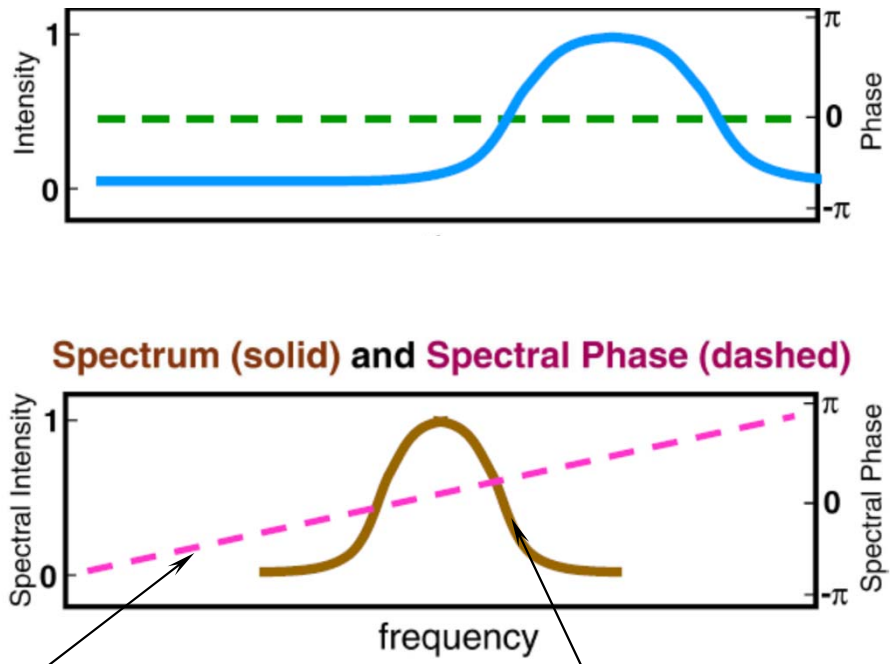
$$|F(u,v)| = \sqrt{F_R^2(u,v) + F_I^2(u,v)}$$

$$\angle F(u,v) = \tan^{-1}\frac{F_I(u,v)}{F_R(u,v)}$$

$$F(u,v) = |F(u,v)|e^{j\angle F(u,v)}$$

Spatial Frequency Domain

Properties



Linear shifting in spatial domain simply adds some linear phase to the pulse

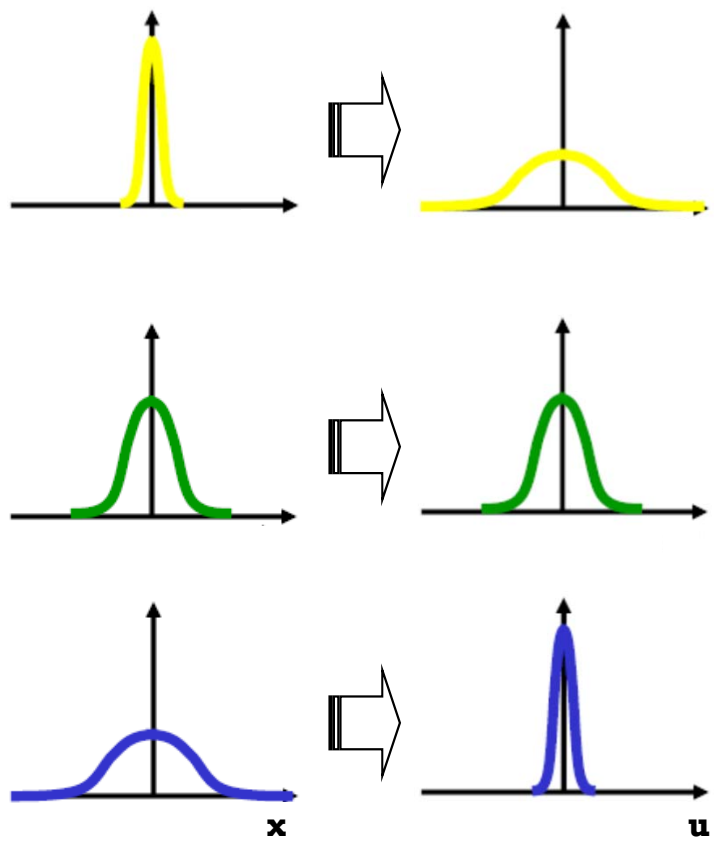
Magnitude is unchanged

Properties of Fourier Transform

- ***Scaling***

$$F\left[f(ax,by)\right]=\frac{1}{ab}F\left(\frac{u}{a},\frac{v}{b}\right)$$

An 1-D example



Spatial domain, f(x) **Fourier Transform, F(u)**

The shorter the pulse, the broader the spectrum !

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$where\, j=\sqrt{-1}$$

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)e^{j2\pi(ux+vy)}dudv$$

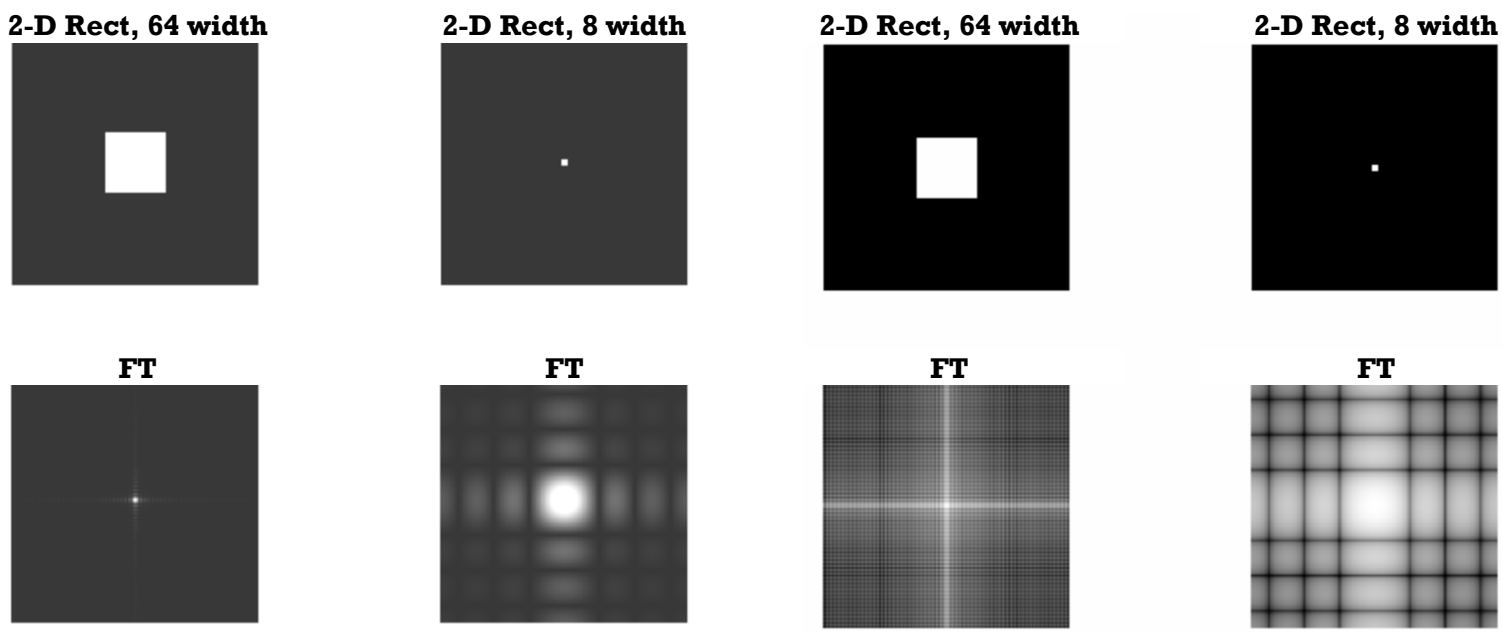
Covered in lecture

Properties of Fourier Transform

- ***Scaling***

$$F\left[f(ax,by)\right]=\frac{1}{ab}F\left(\frac{u}{a},\frac{v}{b}\right)$$

An 2-D example



Linear scale **Log scale**

Covered in lecture

Linear and Shift Invariant Systems Revisited

For a shift-invariant system, the output is the input ***convolved*** with the impulse response function.

$$
\begin{aligned}
g(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)h(x,y,\xi,\eta)\,d\xi\,d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)h(x-\xi,y-\eta)\,d\xi\,d\eta \\
&= f(x,y) * h(x,y)
\end{aligned}
$$

where $h(x,y,\xi,\eta)$ is the response of the system to an delta impulse signal

Covered in lecture

Convolution Theorem

- The convolution of two 2-D functions is defined as

$$f(x,y) * h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\cdot h(x-\xi, y-\eta)\,d\xi\, d\eta$$

- The Fourier transform of the convolution is equal to the product of the individual Fourier transforms:

$$\mathscr{F}\left[f(x,y) * h(x,y)\right] = \mathscr{F}\left[f(x,y)\right]\cdot\mathscr{F}\left[h(x,y)\right]$$

where $\mathscr{F}[\cdot]$ is the Fourier transform operator.

The output from a shift invariant system is therefore

$$g(x,y) = f(x,y) * h(x,y) = \mathscr{F}^{-1}\left\{\mathscr{F}\left[f(x,y)\right]\cdot\mathscr{F}\left[h(x,y)\right]\right\}$$

Covered in lecture

Properties of Fourier Transform

- ***Product***

$$\boldsymbol{F}\left[f(x,y)\cdot g(x,y)\right]=\boldsymbol{F}\left[f(x,y)\right]*\boldsymbol{F}\left[g(x,y)\right]$$

Fourier transform of the product of two functions equals to the convolution of the Fourier transforms of individual function.

- ***The Convolution Theorem***

$$\boldsymbol{F}\left[f(x,y)*g(x,y)\right]=\boldsymbol{F}\left[f(x,y)\right]\cdot\boldsymbol{F}\left[g(x,y)\right]$$

Covered in lecture

Convolution Theorem

Proof:

$$\boldsymbol{F}\left[f(x,y)*g(x,y)\right]$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)\cdot g(x-\xi,y-\eta)\,d\xi\,d\eta\right\}e^{-j2\pi(ux+vy)}\,dx\,dy$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)\cdot g(x',y')\cdot e^{-j2\pi[u(x'+\xi)+v(y'+\eta)]}\cdot dx'\,dy'\right\}d\xi\,d\eta$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)\cdot e^{-j2\pi(u\xi+v\eta)}\cdot g(x',y')\cdot e^{-j2\pi(ux'+vy')}\,dx'\,dy'\right\}d\xi\,d\eta$$

$$=\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(\xi,\eta)\cdot e^{-j2\pi(u\xi+v\eta)}\,d\xi\,d\eta\right]\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(x',y')\cdot e^{-j2\pi(ux'+vy')}\,dx'\,dy'\right]$$

$$=\boldsymbol{F}\left[f(x,y)\right]\cdot\boldsymbol{F}\left[g(x,y)\right]$$

$$\boldsymbol{F}\left[f(x,y)\cdot g(x,y)\right]=\boldsymbol{F}\left[f(x,y)\right]*\boldsymbol{F}\left[g(x,y)\right]$$

Proof:

Note that

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(u,v)e^{j2\pi(ux+vy)}dudv$$

The convolution theorem states:

$$\boldsymbol{F}\left[f(x,y)*g(x,y)\right]=\boldsymbol{F}\left[f(x,y)\right]\cdot\boldsymbol{F}\left[g(x,y)\right]$$

Similarly we can prove that:

$$\boldsymbol{F}^{-1}\left[f(x,y)*g(x,y)\right]=\boldsymbol{F}^{-1}\left[f(x,y)\right]\cdot\boldsymbol{F}^{-1}\left[g(x,y)\right]$$

If we define:

$$F=\boldsymbol{F}^{-1}\left[f(x,y)\right],G=\boldsymbol{F}^{-1}\left[g(x,y)\right],\quad then\quad f(x,y)=\boldsymbol{F}\left[F\right],g(x,y)=\boldsymbol{F}\left[G\right]$$

We can see that

$$\boldsymbol{F}^{-1}\left\{\boldsymbol{F}\left[F\right]*\boldsymbol{F}\left[G\right]\right\}=F\cdot G$$

Therefore

$$\boldsymbol{F}\left[F\cdot G\right]=\boldsymbol{F}\left[F\right]*\boldsymbol{F}\left[G\right]$$

Note that F and G are arbitrary function, so that

we can re - write the above equation as

$$\boldsymbol{F}\left[f(x,y)\cdot g(x,y)\right]=\boldsymbol{F}\left[f(x,y)\right]*\boldsymbol{F}\left[g(x,y)\right]$$

Covered in lecture

Properties of Fourier Transform

- ***Separable product***

$$\textit{if}$$

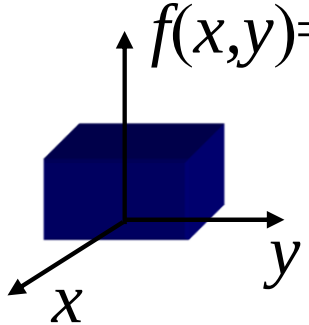
$$f(x,y) = f_1(x) \cdot f_2(y)$$

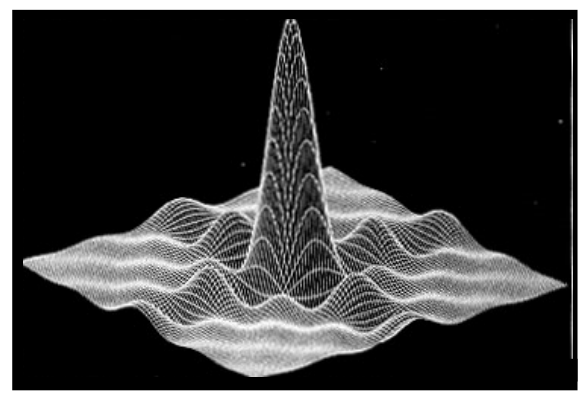
$$\textit{then}$$

$$\boldsymbol{F}_{u,v}\left[f(x,y)\right] = \boldsymbol{F}_u\left[f_1(x)\right] \cdot \boldsymbol{F}_v\left[f_2(y)\right]$$

$$\mathscr{F}\{f(x,y)\} = \mathrm{sinc}(u)\mathrm{sinc}(v)$$

$f(x,y)$=Rect(x)Rect(y)





Symmetry Properties

Expanding the Fourier transform of a function, *f(t):*

$$F(\omega) = \int_{-\infty}^{\infty} [\mathrm{Re}\{f(t)\} + i\,\mathrm{Im}\{f(t)\}]\ [\cos(\omega t) - i\sin(\omega t)]\,dt$$

Expanding more, noting that: $\int_{-\infty}^{\infty} O(t)\,dt = 0$ if $O(t)$ is an odd function

= 0 if Re{*f*(*t*)} is odd **= 0 if Im{*f*(*t*)} is even**

$$F(\omega) = \int_{-\infty}^{\infty} \mathrm{Re}\{f(t)\}\cos(\omega t)\,dt + \int_{-\infty}^{\infty} \mathrm{Im}\{f(t)\}\sin(\omega t)\,dt \leftarrow \mathbf{Re}\{F(\omega)\}$$

= 0 if Im{*f*(*t*)} is odd **= 0 if Re{*f*(*t*)} is even**

$$+\ i\int_{-\infty}^{\infty} \mathrm{Im}\{f(t)\}\cos(\omega t)\,dt - i\int_{-\infty}^{\infty} \mathrm{Re}\{f(t)\}\sin(\omega t)\,dt \leftarrow \mathbf{Im}\{F(\omega)\}$$

Even functions of *w* **Odd functions of *w***

Covered in lecture

Fourier transform of Circularly Symmetric Functions (1)

A function is ***circularly symmetric*** if

$$f_\theta(x,y) = f(x,y), \quad \text{for every } \theta,$$

$$\text{where } f_\theta(x,y) \text{ is a rotated version of } f(x,y).$$

In this case, the function

$$f(x,y) = f(r), \text{ where } r = \sqrt{x^2 + y^2}$$

The Fourier transform is also **real** and **circularly symmetric**.

$$|F(u,v)| = F(u,v) \text{ and } \angle F(u,v) = 0$$

and

$$F(u,v) = F(q), \text{ where } q = \sqrt{u^2 + v^2}$$

Covered in lecture

Fourier Transform of Circularly Symmetric Functions (2)

The Fourier transform of a ***circularly symmetric*** function is called Hankel transform

$$F(q) = 2\pi \int_{-\infty}^{\infty} f(r) J_0(2\pi q r) r dr$$

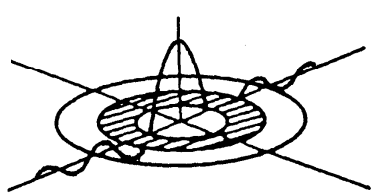
where $J_n(r)$ is the zero - order Bessel function,

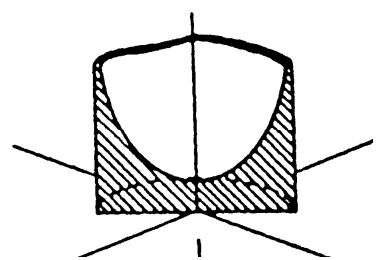
$$J_n(r) = \frac{1}{\pi} \int_0^{\pi} \cos(nr - r\sin\phi) d\phi, \quad n = 0,1,2,...,$$

The inverse Hankel transform is given by

$$f(r) = 2\pi \int_{-\infty}^{\infty} F(q) J_0(2\pi q r) q dq$$

$$\frac{\sin(2\pi a r)}{r}$$





$$\frac{\Pi(q/2a)}{\left(a^2 - q^2\right)^{\frac{1}{2}}}$$

Properties of Fourier Transform (4)

- ***Parseval's Theorem***

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|f(x,y)\right|^2 dudv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|F(u,v)\right|^2 dudv$$

Total energy of a signal of a signal f(x,y) in spatial domain equals its total energy in spatial frequency domain.

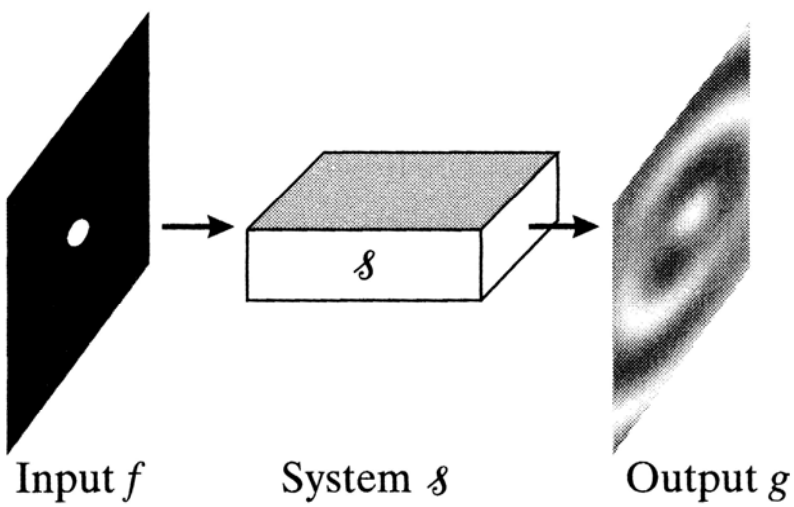
Fourier transform and its inverse is energy preserving!

Convolution Theorem Revisited

The convolution theorem enables one to perform convolution operation as multiplication process in spatial frequency domain.

$$f(x,y) * h(x,y) = \boldsymbol{F}^{-1}\left\{\boldsymbol{F}\left[f(x,y)\right] \cdot \boldsymbol{F}\left[h(x,y)\right]\right\}$$

By using the ***Fast Fourier Transform*** (FFT) algorithms, the convolution operation can be performed very efficiently !! This provide a practical way for modeling linear shift-invariant systems …



$$g(x,y) = f(x,y) * h(x,y)$$

Covered in lecture

System Transfer Function

The Fourier transform of the impulse response function h is called the ***system transfer function***.

$$H(u,v) = \boldsymbol{F}\left[h(x,y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)e^{-2\pi j(ux+vy)}\,dx\,dy$$

$$G(u,v) = F(u,v)H(u,v)$$

Covered in lecture

System Transfer Function

$$G(u,v) = F(u,v)H(u,v)$$

An ***ideal low-pass filter*** is defined as

$$H(u,v) = \begin{cases} 1 & \text{for } \sqrt{\mathrm{u}^2 + \mathrm{v}^2} \le c \\ 0 & \text{for } \sqrt{\mathrm{u}^2 + \mathrm{v}^2} > c \end{cases}$$

c is called the ***cut-off frequency***.

Covered in lecture

System Transfer Function

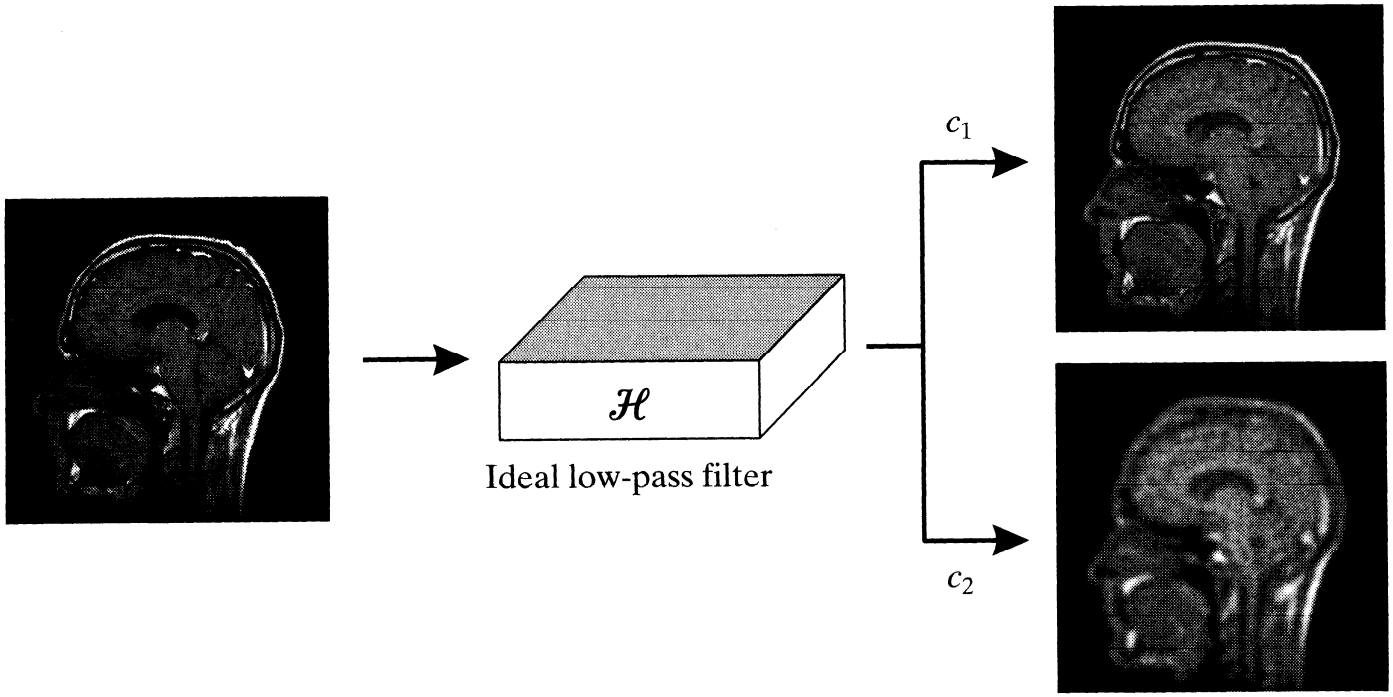


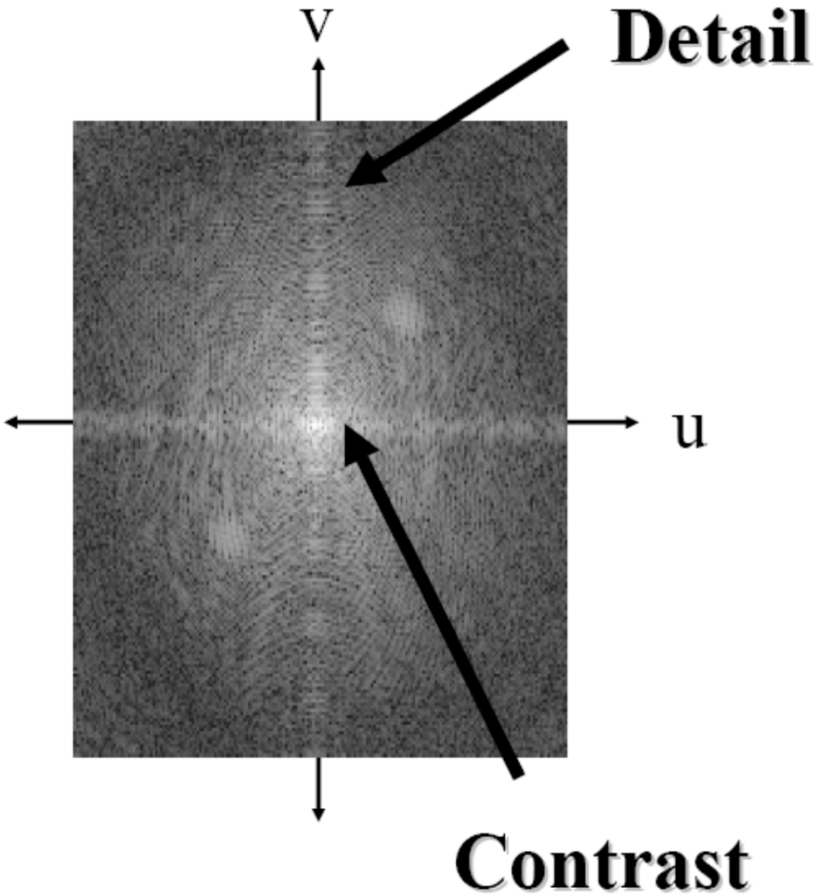
Figure 2.12
The response of an ideal low-pass filter for two values of the cutoff frequency c ($c_1 > c_2$).

Covered in lecture

Spatial Frequency Revisited

Image

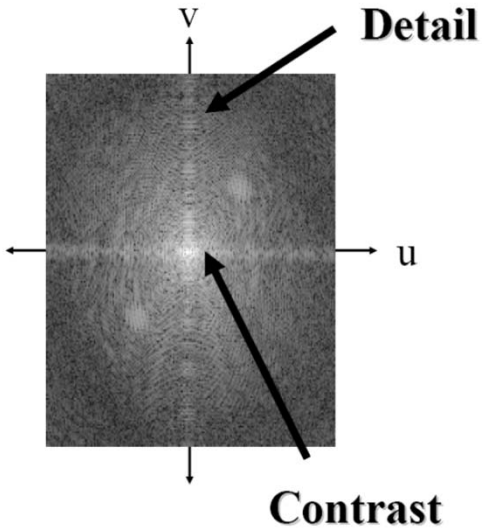
Fourier Space (log magnitude)

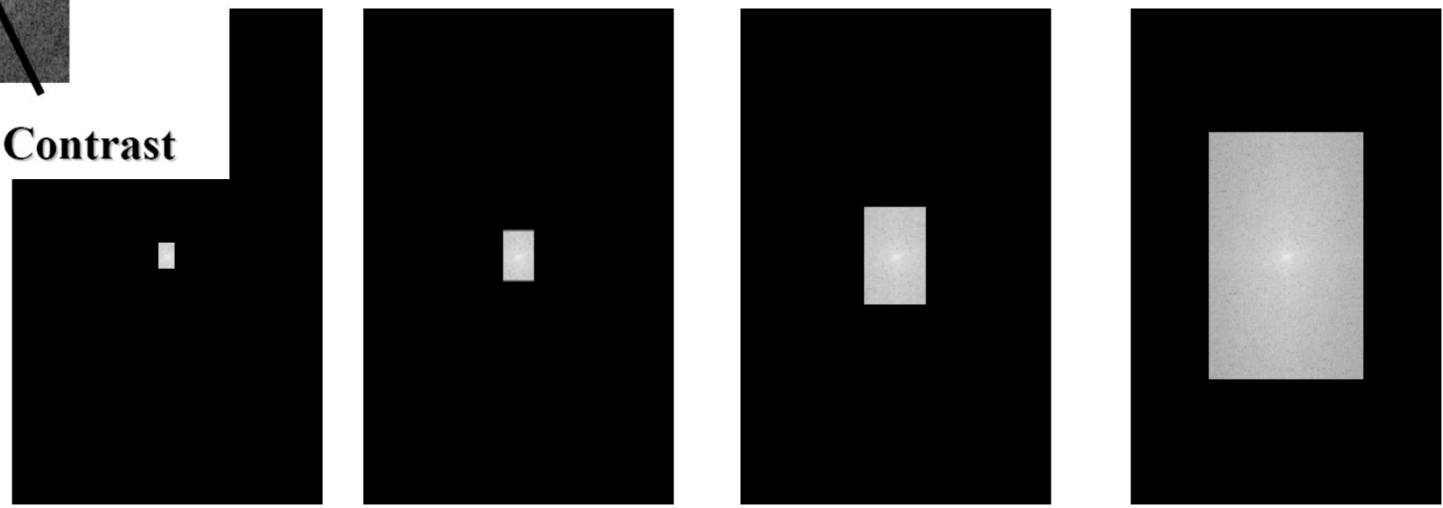


Covered in lecture

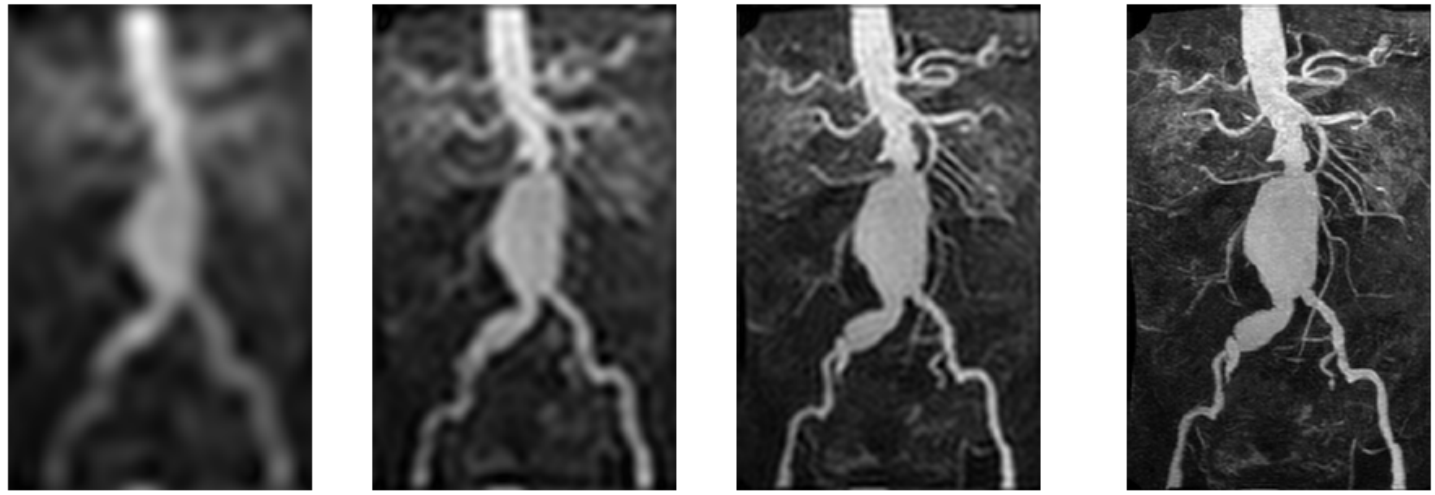
Image

Fourier Space (log magnitude)





5 % 10 % 20 % 50 %

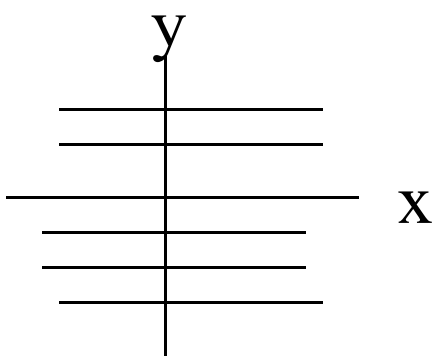


Covered in lecture

Relation Between 1-D and 2-D Fourier Transforms

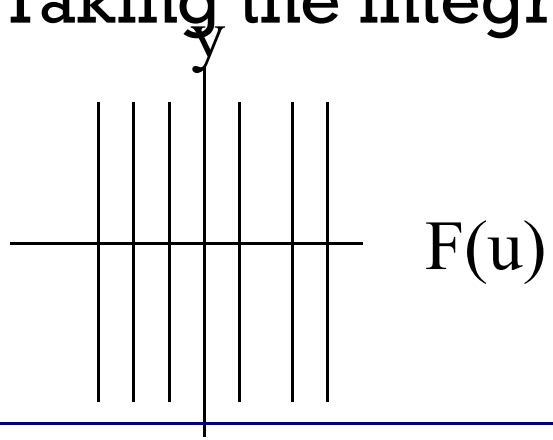
$$\mathrm{F}(u,v) = \int_{-\infty}^{\infty} e^{-i\cdot 2\pi\cdot vy}\left[\int_{-\infty}^{\infty} \mathrm{f}(x,y)\cdot e^{-i\cdot 2\pi\cdot ux}dx\right]dy$$

Rearranging the Fourier Integral,



Taking the integrals along x gives, $\hat{\mathrm{F}}(u,y)$

Taking the integrals of $\hat{\mathrm{F}}(u,y)$ along y gives F(u,v)



$$\mathrm{F}(u,v) = \int_{-\infty}^{\infty} \hat{\mathrm{F}}(u,y)e^{-i\cdot 2\pi\cdot vy}dy$$

Covered in lecture

Basic Fourier Transform Pairs

Basic Fourier transform pairs

Signal	Fourier Transform
1	$\delta(u, v)$
$\delta(x, y)$	1
$\delta(x - x_0, y - y_0)$	$e^{-j2\pi(ux_0+vy_0)}$
$\delta_s(x, y; \Delta x, \Delta y)$	$\mathrm{comb}(u\Delta x, v\Delta y)$
$e^{j2\pi(u_0x+v_0y)}$	$\delta(u - u_0, v - v_0)$
$\sin[2\pi(u_0x + v_0y)]$	$\frac{1}{2j}[\delta(u - u_0, v - v_0) - \delta(u + u_0, v + v_0)]$
$\cos[2\pi(u_0x + v_0y)]$	$\frac{1}{2}[\delta(u - u_0, v - v_0) + \delta(u + u_0, v + v_0)]$
$\mathrm{rect}(x, y)$	$\mathrm{sinc}(u, v)$
$\mathrm{sinc}(x, y)$	$\mathrm{rect}(u, v)$
$\mathrm{comb}(x, y)$	$\mathrm{comb}(u, v)$
$e^{-\pi(x^2+y^2)}$	$e^{-\pi(u^2+v^2)}$