MODULE 1: INTRODUCTION

Data Analysis in MATLAB
I. Introducing MATLAB
What is MATLAB?

MATrix LABoratory

- A calculator
- A programming language
- A modeling platform
- An application developer
- A financial modeling tool
- A signal processing platform
- An optimization tool
- A teaching tool
- A bioinformatics framework
- A visualization suite
- A symbolic math tool
- A graphical plotter
- A data analysis toolbox
What is MATLAB?

“A numerical computing environment and fourth-gen programming language developed by MathWorks”
- wikipedia.org

- Initially developed by Cleve Moler, U. NM in 1970’s

- Now a commercial for-profit product developed by Mathworks (Natick, MA)
Why MATLAB?

- Powerful interpreted language for math manipulations
- The “right” tool for data exploration & analysis
- Excellent, well-documented, and easy-to-use interface
- Many specialized and powerful toolboxes (stats, controls, bioinformatics, optimization,...)
- High-quality, customizable, publication-standard graphics
- Simple interfaces, but can access “under the hood”
- Freely available site license to all UIUC personnel (need VPN access for off-campus IP addresses)
Why not MATLAB?

- Interpreted language, and therefore slow...

- ...but Matlab compiler can generate stand-alone executables
What alternatives are there?

- Commercial competitors:
  Maple, Mathematica, IDL

- Open source (free) alternatives:
  GNU Octave, Python, Scilab, FreeMat

- Online GNU Octave server (useful in a pinch):
  http://www.online-utility.org/math/math_calculator.jsp
Can’t I use Excel?

The short answer

NO.
Can’t I use Excel?

The longer answer

It is a goal of this course to develop competency and proficiency in scientific software, including Matlab.

All analyses and plots submitted as part of the homework projects are expected to be done in Matlab.
Can’t I use Excel?

The real answer

- Excel is a terrific tool for quick and dirty data analysis, data storage, and spreadsheeting
- It lacks math firepower for sophisticated data analysis
- Analysis is invariably less efficient and clunkier than Matlab
- Graphics are not of publication quality
II. Basics
MATLAB must be loaded from the terminal, it is not currently available through the Applications menu.

Bash ninjas may add `module load matlab/R2015a` to their `.bash_profile`.

```bash
[alf@dcl-l440-15 ~]$ module avail matlab
Rebuilding cache, please wait ... (not written to file) done

/etc/modulefiles/software
matlab/R2009a   matlab/R2013b   matlab-research/R2011a (D)
matlab/R2009b   matlab/R2014a   matlab-research/R2013a
matlab/R2010b   matlab/R2014b   matlab-research/R2014b
matlab/R2011a (D) matlab/R2015a   matlab-research/R2009b

Where:
D: Default Module

Use "module spider" to find all possible modules.
Use "module keyword key1 key2 ..." to search for all possible modules matching any of the "keys".

[alf@dcl-l440-15 ~]$ module load matlab/R2015a
[alf@dcl-l440-15 ~]$ matlab
```
MATLAB is a high-level, interpreted programming language.

The main interface is through typing text into the command line or executing functions/scripts.
The MATLAB interface
The MATLAB interface
The MATLAB interface

- Can access GUI or CLI via EWS remote login

**CLI:**

```bash
ssh <username>@remlnx.ews.illinois.edu
module load matlab/R2015a
matlab -nodesktop -nодisplay
```

**GUI:**

```bash
ssh -Y <username>@remlnx.ews.illinois.edu
module load matlab/R2015a
matlab
```

*(X-forwarding over slow connection is impractical)*
The MATLAB ethos

Simplicity and versatility

- Naturally based around vectors, matrices and tensors
- Weakly typed
- Dynamically typed
- Structures and classes (OOP) supported
- User defined and built-in functions
- Powerful scripting and visualization interfaces
Creating variables

MATLAB window showing:

```matlab
>> a=27
a =
    27
>> b=[1 3 5 7 9]
b =
    1   3   5   7   9
>> c=[1 2 3; 6 9 12; 33 66 99]
c =
    1   2   3
    6   9  12
   33  66  99
```
Variable names

- Use short, descriptive names
  - index ✓
  - sideLength ✓
  - temperature ✓
  - theSizeOfTheBoxAtTimeZero ✗
  - fajfoiunejhiuhnnjkfa ✗

- Naming rules:
  - < 63 characters
  - can't start with a number
  - not a keyword

- 17 keywords:
  - for
  - function
  - otherwise
  - try
  - break
  - end
  - return
  - switch
  - catch
  - if
  - elseif
  - continue
  - global
  - while
  - case
  - else
  - persistent
Calling functions
Writing functions

MATLAB code snippet:

```matlab
%%load('G.mat','G1','G2')
G1=[
0 1 0 1 0 0 0
1 0 1 0 0 0 0
0 0 1 0 0 0 0
0 0 0 1 0 0 0
0 0 0 0 1 0 0
];
G2=[
0 0 1 0 0 0 0
1 0 0 1 0 0 0
0 0 0 0 1 0 0
0 0 0 0 0 1 0
];
n1=size(G1,1);
if size(G1,2)~=n1
    error('G1 is not square');
end
n2=size(G2,1);
if size(G2,2)~=n2
    error('G2 is not square');
end
H1=zeros(n1);
H2=zeros(n2);
for i=1:n1
    for j=1:n1
        if i==j
            H1(i,j)=1;
        elseif G1(i,j)==1
            H1(i,j)=exp(-G1(i,j));
        end
    end
end
for i=1:n2
    for j=1:n2
        if i==j
            H2(i,j)=1;
        elseif G2(i,j)==1
            H2(i,j)=exp(-G2(i,j));
        end
    end
end
```

14 usages of 'n1' found
The MATLAB documentation is excellent
Getting help

Google Search:

**matlab bar**

About 5,680,000 results (0.16 seconds)

**Bar graph - MATLAB bar - MathWorks**

www.mathworks.com/help/matlab/ref/bar.html

This MATLAB function draws one bar for each element in Y.

**Bar chart - MATLAB bar, barh - MathWorks**

www.mathworks.com/help/finance/barbarh.html

tsobj. Financial time series object. width. Width of the bars and separation of bars within a group. (Default = 0.8.) If width is 1, the bars within a group touch one...

**Bar and Area Graphs - MATLAB & Simulink - MathWorks**


View results over time, comparing results, and displaying individual contribution to a total amount.

**MATLAB Plot Gallery - Vertical Bar Plot: Vertical Bar Plot - MathW...**

www.mathworks.com/matlabcentral/...matlab...bar.../Vertical_Bar_Plot.ht...

This is an example of how to create a vertical bar chart in MATLAB®. Read about the bar function in the MATLAB® documentation. Go to MATLAB Plot Gallery

**MATLAB Plot Gallery - Stacked Bar Chart: Stacked Bar Chart - Fil...**

www.mathworks.com/matlabcentral/...matlab...bar.../Stacked_Bar_Chart....

This is an example of how to create a stacked bar chart in MATLAB®. Read about the bar function in the MATLAB® documentation. Go to MATLAB Plot Gallery
III. Data Visualization
Let’s together run through a number of common (but powerful) data visualization and exploration techniques.

To make things concrete, we shall analyze the \( \sigma \) and \( \varepsilon \) parameters for a number of water models used in molecular dynamics simulations.
Load data

- Copy the file water_models.csv from /class/mse404pla
- csv = comma separated values
- Can open in Excel:
Load data

(i) Copy and paste


$$\epsilon = \begin{bmatrix} 0.65 \\ 0.65 \\ 0.65 \\ 0.6364 \\ 0.6368 \\ 0.6 \end{bmatrix}$$
(ii) textscan

```matlab
>> fid=fopen('water_models.csv','rt');
>> C=textscan(fid,'%*s %f %f','headerlines',3,'delimiter','','');
>> fclose(fid);
>> sigma=C{1};
>> epsilon=C{2};
```
>> scrsz = get(0,'ScreenSize');
>> figure('Position',[0 scrsz(4)/2 scrsz(3)/2 scrsz(4)/2])
>> plot(sigma,epsilon,'ro-')

>> saveas(gcf,'myFigure','fig')
>> saveas(gcf,'myFigure','jpg')
I. plot

```matlab
>> set(gca,'fontsize',18)
>> set(gca,'color','w')
>> ylabel('$\epsilon / \text{kJ/mol}$','fontsize',22)
>> xlabel('$\sigma / \text{Angstoms}$','fontsize',22)
>> xlim([3 4])
>> set(gca,'xtick',3:0.25:4)
>> set(gca,'xticklabel',{'3.00','3.25','3.50','3.75','4.00'})
>> ylim([0 1])
>> set(gca,'ytick',0:0.25:1)
>> set(gca,'yticklabel',{'0.00','0.25','0.50','0.75','1.00'})
```
2. scatter

```matlab
>> scatter(sigma,epsilon,55,epsilon,'filled')
>> colorbar
>> set(gca,'fontsize',18)
>> xlabel('$\sigma$ / Angstoms','fontsize',22)
>> ylabel('$\epsilon$ / kJ/mol','fontsize',22)
>> set(gcf,'color','w')
```
3. hist

```matlab
>> [count,bins]=hist(sigma,3:0.1:4);
>> bar(bins,count)
>> xlabel('\sigma / Angstoms','fontsize',22)
>> ylabel('count / -','fontsize',22)
>> set(gcf,'color','w')
>> set(gca,'fontsize',18)
```
4. hist3

```matlab
>> data=cat(2,sigma,epsilon);
>> bins_sigma=3:0.2:4;
>> bins_epsilon=0:0.25:1;
>> bins=cell(2,1);
>> bins{1}=bins_sigma; bins{2}=bins_epsilon;

>> hist3(data,bins)
>> xlabel('\sigma / Angstoms', 'fontsize',22)
>> ylabel('\epsilon / kJ/mol', 'fontsize',22)
>> zlabel('count / -', 'fontsize',22)
>> set(gca,'fontsize',18)
>> set(gcf,'color','w')
```
>> scatterhist(sigma, epsilon, 'NBins', [20, 20])
6. surf

>> [count,bins] = hist3(data,bins)

>> bins_X = bins{1};
>> bins_Y = bins{2};
>> [X,Y]=meshgrid(bins{1},bins{2})

>> surf(X,Y,count')
>> colorbar
>> xlabel('
\sigma / \text{Angstoms}',... 'fontsize',22)
>> ylabel('
\epsilon / \text{kJ/mol}',... 'fontsize',22)
>> zlabel('count / -',... 'fontsize',22)
>> set(gca,'fontsize',18)
>> set(gcf,'color','w')

What happens when you replace surf with mesh / meshc?
IV. Data Analysis
Now let’s consider a number of useful data analysis tools and statistical tests.
null hypothesis

**noun**

(in a statistical test) the hypothesis that there is no significant difference between specified populations, any observed difference being due to sampling or experimental error.

- By default, we assume that the null hypothesis is true.

- We apply statistical tests to assess whether there is sufficient evidence to reject the null hypothesis.

- We reject the null hypothesis if the observed relationship in the data is sufficiently unlikely to have arisen by chance if the null hypothesis were true ($p < \alpha = 0.05, 0.01$).
1. Pearson’s correlation coefficient ($r$)

**Purpose**
Measure of the **linear** correlation between two variables. Limited to range $[-1, 1]$.

**Theory**

\[
\rho_{X,Y} = \frac{\text{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X \sigma_Y}
\]

- Fails to uncover **nonlinear** relationships.
- Use Spearman corr coeff for **rank** correlation (monotonicity)

[Link to Wikipedia Correlation Examples](http://en.wikipedia.org/wiki/File:Correlation_examples2.svg)
If x & y are uncorrelated Gaussian distributions, Pearson's r follows a Student's t-distribution with (n-2) dof.

Using this distribution, we ask: “what is the probability that the observed Pearson’s r value arose by chance given that the true correlation is zero?”

95% CI on Pearson’s r or “what is the expected range of r given our finite data sample?”
2. Permutation Test

- **Purpose**
  A non-parametric hypothesis test.

Without assuming a distribution, answers: “What is the probability the observed result occurred by chance?”

- **Theory**
  We perform random shuffles of the data and compute the test statistic.

The p-value is the proportion of shuffled test statistics that are greater than the observed value.
2. Permutation Test

Practice

```matlab
>> R_obs=corrcoef(sigma,epsilon);
   r_obs = R_obs(1,2)

   r_obs =
       0.2441

>> n=length(sigma);
nTests=1000;
r_array=zeros(nTests,1);

   for i=1:nTests
      idx=randperm(n);
      R = corrcoef(sigma,epsilon(idx));
      r_array(i) = R(1,2);
   end

   p_value = sum(abs(r_array)>abs(r_obs))/nTests
   p_value =
       0.2630
```
3. Bootstrap

- **Purpose**
  Non-parametric estimate of test statistic confidence interval

  Without assuming a distribution, answers: “*Given our finite sample, what range of test statistics might we have seen?*”

- **Theory**
  Our data typically represents a finite sample from a large population (e.g., human heights, component lifetimes, etc.)

  Different samples of $n$ data points produce different results

  Bootstrap simulates different samples by resampling with replacement
3. Bootstrap

**Practice**

```matlab
>> n = length(sigma);
nTests = 1000;
replacement=true;
r_array = zeros(nTests,1);

for i=1:nTests
    idx = randsample(n,n,replacement);
    R = corrcoef(sigma(idx),epsilon(idx));
    r_array(i) = R(1,2);
end

[count,bins] = hist(r_array,100);
prob = count/sum(count);
cumProb = cumsum(prob);

alpha=0.05;
[~,idx_lo] = find(cumProb<=alpha,1,'last');
CI_lo = bins(idx_lo);
[~,idx_hi] = find(cumProb>=(1-alpha),1,'first');
CI_hi = bins(idx_hi);

fprintf('

');
fprintf('RESULT: %.0f%% CI = [%.2f,%.2f]
',(1-2*alpha)*100,CI_lo,CI_hi);

RESULT: 90% CI = [-0.48,0.58]
```
4. Multiple Linear Regression

- **Purpose**
  Attempt to recover predictor of a scalar dependent variable, $y$, as a linear combination of independent variables, $x$

- **Theory**
  **MLR model:**
  $y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \ldots + \beta_m x_{im} + \epsilon_i = \vec{x}_i^T \vec{\beta} + \epsilon_i$
  $\vec{y} = X\vec{\beta} + \vec{\epsilon}$

  **OLS estimate:**
  $\vec{\beta}^* = \arg\min_{\vec{\beta}} ||\vec{\epsilon}|| = \arg\min_{\vec{\beta}} ||\vec{y} - X\vec{\beta}||$

  **Assumptions:**
  linear, independent $x$, homoscedastic, no multicollinearity
4. Multiple Linear Regression

**Practice**

```
>> y=sigma;
>> X=[epsilon ones(length(epsilon),1)];
>> [b,bint,r,rint,stats] = regress(y,X)
```

\[ b = \]
\[
\begin{align*}
0.2259 & \quad \beta \\
3.0417 & \\
\end{align*}
\]

\[ bint = \]
\[
\begin{align*}
-0.2050 & \quad \beta 95\% CI \\
2.7373 & \quad 3.3460 \\
\end{align*}
\]

\[ r = \]
\[
\begin{align*}
-0.0225 & \\
-0.0225 & \\
-0.0225 & \\
-0.0348 & \\
-0.0349 & \\
0.0568 & \\
-0.0344 & \\
-0.0312 & \\
-0.0344 & \\
-0.0742 & \\
-0.0579 & \\
-0.0524 & \\
-0.0805 & \\
0.2792 & \\
0.4417 & \\
-0.0571 & \\
0.0035 & \\
-0.0133 & \\
-0.0729 & \\
-0.1130 & \\
\end{align*}
\]

\[ rint = \]
\[
\begin{align*}
-0.2955 & \quad 0.2505 \\
-0.2955 & \quad 0.2505 \\
-0.2955 & \quad 0.2505 \\
-0.2955 & \quad 0.2505 \\
-0.3071 & \quad 0.2374 \\
-0.3072 & \quad 0.2373 \\
-0.2126 & \quad 0.3261 \\
-0.3071 & \quad 0.2382 \\
-0.3046 & \quad 0.2422 \\
-0.3071 & \quad 0.2382 \\
-0.3323 & \quad 0.1839 \\
-0.3278 & \quad 0.2121 \\
-0.3226 & \quad 0.2179 \\
-0.3275 & \quad 0.1665 \\
0.0564 & \quad 0.5020 \\
0.3080 & \quad 0.5754 \\
-0.3163 & \quad 0.2020 \\
-0.2703 & \quad 0.2773 \\
-0.2308 & \quad 0.2042 \\
-0.3442 & \quad 0.1984 \\
-0.3801 & \quad 0.1542 \\
\end{align*}
\]

\[ stats = \]
\[
\begin{align*}
0.0596 & \\
1.2042 & \\
0.2862 & \\
0.0170 & \\
\end{align*}
\]

\[ s^2 = \sqrt{\text{RMSE}} \quad \text{(error variance)} \]

\[ R^2 = 1 - \frac{\text{SSE}}{\text{TSS}} \]

\[ \text{F-stat} \quad \text{p-value} \]
4. Multiple Linear Regression

The F-test assesses whether the fitted regression model gives a statistically significant better fit to the data than simply describing the data by its mean.

Model 1: Mean*
\[ \hat{y} = \bar{\varepsilon} \]

- # params = 1
- dof = n-1

Model 2: Regression*
\[ \hat{y} = \mathbf{X}\beta + \varepsilon \]

- # params = k
- dof = n-k

\[ F = \frac{RSS_1 - RSS_2}{RSS_2/n - dof_2} \]

follows an F-distribution \( F(\text{dof}_2 - \text{dof}_1, n - \text{dof}_2) \) under null hypothesis that Model 2 is not better than Model 1.

p-value = probability of observing this large (or larger) F-value by chance if the null hypothesis is true

assert significance (yes/no) by specifying a significance cutoff alpha (usually alpha = 0.05)

*Model 1 must be a restriction / specialization of Model 2
REGRESSION OR CORRELATION?

Linear regression and correlation are similar and easily confused. In some situations it makes sense to perform both calculations. Calculate linear correlation if you measured both X and Y in each subject and wish to quantify how well they are associated. Select the Pearson (parametric) correlation coefficient if you can assume that both X and Y are sampled from Gaussian populations. Otherwise choose the Spearman nonparametric correlation coefficient. Don't calculate the correlation coefficient (or its confidence interval) if you manipulated the X variable.

Calculate linear regressions only if one of the variables (X) is likely to precede or cause the other variable (Y). Definitely choose linear regression if you manipulated the X variable. It makes a big difference which variable is called X and which is called Y, as linear regression calculations are not symmetrical with respect to X and Y. If you swap the two variables, you will obtain a different regression line. In contrast, linear correlation calculations are symmetrical with respect to X and Y. If you swap the labels X and Y, you will still get the same correlation coefficient.
5. Akaike Information Criterion (AIC)

**Purpose**
Model discrimination criterion, “what model should I choose?”
Trade-off between goodness-of-fit and model complexity

“With four parameters I can fit an elephant, and with five I can make him wiggle his trunk”

-John von Neumann

5. Akaike Information Criterion (AIC)

Theory

Information theoretical measure
Estimate of information loss relative to “true” model
Penalizes more parameters and poor fits

\[
AIC = 2k - 2\ln(L)
\]

where:
- \( k \) = number of model parameters
- \( L \) = likelihood of model given data

For i.i.d. normally distributed errors

\[
AIC = 2k + n\ln\left(\frac{RSS}{n}\right)
\]

where:
- \( n \) = number of data points
- RSS = residual sum of squares

Compute AIC for various models and choose \( \min(AIC) \)

5. Akaike Information Criterion (AIC)

**Practice**

Use AIC to discriminate between regression models:

(i) \( \sigma = \beta_1 \varepsilon + c \)

(ii) \( \sigma = \beta_2 \varepsilon^2 + \beta_1 \varepsilon + c \)

(iii) \( \sigma = \beta_3 \varepsilon^3 + \beta_2 \varepsilon^2 + \beta_1 \varepsilon + c \)

```matlab
>> n=length(sigma);
>> k=[2 3 4];
>> RSS=[r1'*r1 r2'*r2 r3'*r3];
>> for i=1:3
    AIC(i) = 2*k(i) + n*log(RSS(i)/n);
end
>> AIC
AIC =
-83.6249  -82.6224  -85.9937
```
5. Akaike Information Criterion (AIC)

But beware!

AIC measures only **relative**, not **absolute** model quality!

```
>> r2 = [stats1(1) stats2(1) stats3(1)]

r2 =

   0.0596    0.1032    0.3056
```
6. Cross Validation

- **Purpose**
  Empirical assessment of model performance on “new” data
  Alternative to AIC for model discrimination
  Quantitative assessment of model over-fitting

- **Theory**
  MSE measured over training data over optimistic prediction of performance on new data - “in-sample MSE”

Break data into training and validation sets, the CV-MSE or “out-of-sample MSE” better measure of model performance

Common splits: **k-fold CV**
  **leave-one-out CV (LOO-CV)**
6. Cross Validation

**Practice**

Use LOO-CV to discriminate between regression models:

(i) \( \sigma = \beta_1 \varepsilon + c \)
(ii) \( \sigma = \beta_2 \varepsilon^2 + \beta_1 \varepsilon + c \)
(iii) \( \sigma = \beta_3 \varepsilon^3 + \beta_2 \varepsilon^2 + \beta_1 \varepsilon + c \)

<table>
<thead>
<tr>
<th>MSE</th>
<th>LOOCV-MSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>0.0154</td>
</tr>
<tr>
<td>(ii)</td>
<td>0.0147</td>
</tr>
<tr>
<td>(iii)</td>
<td>0.0114</td>
</tr>
</tbody>
</table>

```matlab
>> n=length(sigma);
y=sigma;

% standard OLSR
X=[epsilon,ones(length(epsilon),1)];
[b,~,r,~,stats] = regress(y,X);
MSE = mean(r.^2);

% LOO-CV
CV_MSE_array = nan(n,1);
for p=1:n    % running over LOO
    % building LOO independent variables
    idx = setdiff(1:n,p);
    X = [epsilon(idx),ones(length(idx),1)];
    % performing regression
    [b,~,r,~,stats] = regress(y(idx),X);
    % computing CV_MSE
    X_solo = [epsilon(p),1];
    CV_MSE_array(p) = (y(p) - X_solo*b).^2;
end
CV_MSE = mean(CV_MSE_array);

>> fprintf('MSE = %f, LOOCV-MSE = %f
',MSE,CV_MSE)
MSE = 0.015412, LOOCV-MSE = 0.021340
```
6. Cross Validation

The models immediately begin to overfit with increasing complexity (indicative of a poor modeling paradigm)

Select model using minimum or knee in MSE and CV-MSE curves
7. Student’s t-test

- **Purpose**
  - Are two data sets significantly different?
  - or Are two data sets drawn from same underlying dist’n?

- **Theory**
  Assumes:
  - two data sets are independent
  - each set normally distributed if scaling term were known
  - small (n < 30) sample sizes

For large sample sizes, use **Z-test**
For non-normally distributed data use **Mann-Whitney**
7. Student’s t-test

**Theory**

\[ t = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \]

\[ dof = \frac{\left( \frac{s_1^2}{n_1} + \frac{s_2^2}{n_2} \right)^2}{\frac{s_1^2}{n_1 - 1} + \frac{s_2^2}{n_2 - 1}} \]

determine significance at level $\alpha$ from Student’s t-distribution with $dof$
7. Student’s t-test

**Practice**

Determine if TIP4P water model sigma parameters follow a different distribution from the rest of the models.

```matlab
>> sigma_TIP4P = sigma(8:13);
>> sigma_rest = sigma(setdiff(1:length(sigma),8:13));
>> alpha = 0.05;
>> [h,p,ci,stats] = ttest2(sigma_TIP4P,sigma_rest,alpha,'both','unequal')

h =
    0

p =
    0.2043

ci =
    -0.1384
    0.0324

stats =
    tstat: -1.3311
    df: 14.0976
    sd: [2x1 double]

>> t_crit = tinv(alpha/2,stats.df)

t_crit =
    -2.1434
```

- **split data & specify significance**
- **perform t-test**
  - `ttest2` = 2 independent samples
  - `'both'` = means not equal
  - `'unequal'` = variances not equal
  - `h = 1` => reject null hypothesis
  - `h = 0` => accept null hypothesis
  - `p` = p-value under null hypothesis of observed or more extreme t-value
  - `ci` = confidence interval at alpha critical t-value to reject null hypothesis