Lect 9: Inverse kinematics and numerical differentiation

How are segment angles and joint angles calculated?
What are finite differences and how does it relate to “numerical noise”?
What 3 methods are used in numerical differentiation?

Monday -> Lab 1 (CoP) due
HW2 -> IRB (due Feb 17)
RECAP

Static balance
  ➔ COP, postural control

Filtering & signal processing
  Statistical tests

+ Phases of the Gait cycle
Movement:
- Quantifying gait cycle
- Determinants of gait (keep COM moving efficiently in a smooth forward trajectory)

Muscle activation

\[ T = \text{Joint torques} \]
\[ \theta = \text{Joint angles} \]

\[ \text{inverse dynamics} \]

MOVE!

A(partial):
- Segment positions
- External loads

\( \theta(1) \quad \theta'(1) \)

Walking: Ground reaction force
Angle calculations

Model body as multiple rigid segments

Assume joints are pin joints

⇒ joint centers ⇒ come from mocap
- skin motion artifact!

Segment lengths are constant
**Segment angles:** Absolute angle (relative to global coordinates, for example, +x axis) based on joint centers

\[
S_1 = (S_x, S_z) \\
H_1 = (H_x, H_z)
\]

\[
\theta = \tan^{-1}\left(\frac{\Delta z}{\Delta x}\right)
\]

- \( \Delta z \) defined as + CCW from horizontal
- Use prox-dist: \( \Delta z = S_z - H_z \) \\
  \( \Delta x = S_x - H_x \)

**Matlab:** \( \theta = \text{atan2}(\Delta z, \Delta x) \) (radians!)

If \( \Delta z < 0 \) \( \Rightarrow \) \( \theta = \uparrow + 2\pi \)
Let's work an example

\[ \phi_{\text{segment}} = \tan^{-1} \left( \frac{\Delta z}{\Delta x} \right) \]

\[ = \tan^{-1} \left( \frac{pz - dz}{px - dx} \right) \]

where \( p \): proximal, \( d \): distal

<table>
<thead>
<tr>
<th>Joint center</th>
<th>X coord (m)</th>
<th>Z coord (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ankle</td>
<td>2.4</td>
<td>0.5</td>
</tr>
<tr>
<td>Hip</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Knee</td>
<td>2.5</td>
<td>0.75</td>
</tr>
<tr>
<td>Shoulder</td>
<td>2</td>
<td>1.75</td>
</tr>
<tr>
<td>Toe</td>
<td>2.7</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Segment

trunk

shoulder

joints

segment angles

hip

θ trunk

thigh

θ thigh

knee

θ shank

shank

θ foot

ankle

foot

toe
Angular velocities + accelerations

velocity = \omega = \frac{\Delta \Theta_{	ext{hip}}}{\Delta t}

Acceleration = \alpha = \frac{\Delta \omega}{\Delta t}
Now, let’s add the forces → kinetics

- Biceps brachii muscle example

- Moment generating capacity of a muscle
Numerical differentiation

Know: \( x(t) \equiv \text{position [m]} \)

Want:
Recall Taylor Series Expansion

\[ f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots. \]

Based on Taylor Series Expansion

\[ x_{n+1} = x_n + \frac{d(x_n)}{dt} \Delta t + \frac{1}{2} \frac{d^2(x_n)}{dt^2} (\Delta t)^2 + O(\Delta t^3) + O(\Delta t^4) + \cdots \]

or

\[ x_{n-1} = x_n - \frac{d(x_n)}{dt} \Delta t + \frac{1}{2} \frac{d^2(x_n)}{dt^2} (\Delta t)^2 - O(\Delta t^3) + O(\Delta t^4) + \cdots \]
How to calculate velocity if we know position?

Three options:

- Euler’s Method (Forward Difference Method)
  \[
  \frac{d(x_n)}{dt} = \frac{x_{n+1} - x_n}{\Delta t}
  \]

- Backward Difference Method
  \[
  \frac{d(x_n)}{dt} = \frac{x_n - x_{n-1}}{\Delta t}
  \]

- Three-Point Formula (Centered-Difference Formula)
  \[
  \frac{d(x_n)}{dt} = \frac{1}{2} \left[ \frac{x_{n+1} - x_{n-1}}{\Delta t} \right]
  \]
Euler's Method (Forward Difference Method)
Backward Difference Method
Three-Point Formula (Centered-Difference Formula)
Second Order Derivative: accelerations

- Based on Taylor Series Expansion

\[ x_{n+1} = x_n + \frac{d(x_n)}{dt} \Delta t + \frac{1}{2} \frac{d^2(x_n)}{dt^2} (\Delta t)^2 + O(\Delta t^3) + O(\Delta t^4) + \ldots \]

or

\[ x_{n-1} = x_n - \frac{d(x_n)}{dt} \Delta t + \frac{1}{2} \frac{d^2(x_n)}{dt^2} (\Delta t)^2 - O(\Delta t^3) + O(\Delta t^4) + \ldots \]

combine

\[ x_{n+1} + x_{n-1} \approx 2x_n + \frac{d^2(x_n)}{dt^2} (\Delta t)^2 + 2O(\Delta t^4) + \ldots \]

rearrange to get:

\[ \frac{d^2(x_n)}{dt^2} = \frac{x_{n+1} - 2x_n + x_{n-1}}{(\Delta t)^2} \]