ECE 588 – Electricity Resource Planning

3. Markov Models for Reliability Evaluation

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A stochastic process is a collection of random variables indexed by some parameter $t$; we denote a stochastic process by the set $\{X_t : t \in \mathcal{T}\}$.

The c.d.f. $F(\cdot)$ is a function of the values of $X_{\xi}$ for $\xi < t$, the past history of the process:

$$F_{X_t}(x) = P\{X_t \leq x\} \leftarrow \text{a function of } X_{\xi} \text{ for } \xi < t$$

The terms stochastic process and random process are synonymous.
A Markov process is a stochastic process with the property that \( \forall n = 1, 2, \ldots \) and for any indices \( t_1 < t_2 < \ldots < t_n \)

\[
P\left\{ X_{t_n} \leq x_n \mid X_{t_{n-1}}, X_{t_{n-2}}, \ldots, X_{t_1} \right\} = P\left\{ X_{t_n} \leq x_n \mid X_{t_{n-1}} \right\}
\]

or equivalently,

\[
P\left\{ X_{t_n} \leq x_n \mid X_{t} \forall t \leq t_{n-1} \right\} = P\left\{ X_{t_n} \leq x_n \mid X_{t_{n-1}} \right\}
\]
In other words, the c.d.f. $F_{X_{tn}}(\cdot)$ of a Markov process depends only on $X_{t_{n-1}}$ and not on its previous history, i.e., the value of $X_{t_{n-1}}$ captures the entire past history of the process $\{X_t : t \in \mathcal{T}\}$ for $t \leq t_{n-1}$.

A Markov process with a discrete state space, i.e., the values of $X_t$, at any time $t$ are discrete, is called a Markov chain.
MARKOV CHAIN

- We consider a Markov chain with a continuous parameter \( t \)
  - \( t \in \mathbb{R}_+ \) covers some interval of interest
  - \( X(t) \) can take on only a finite number \( m \) of discrete values, which we denote by \( i_1, i_2, \ldots, i_m \)

- We define the transitional probability \( p_{ji}(t, h) \) to be the probability of transition from state \( i \) at time \( t \) to state \( j \) at time \( t + h \), with

\[
p_{ji}(t, h) \triangleq P \left\{ X(t + h) = j \mid X(t) = i \right\}
\]
We call a Markov chain *homogeneous* if \( p_{ji}(t, h) \) is independent of \( t \) but dependent only on \( h \):

\[
P \{ X(t + h) = j \mid X(t) = i \} = p_{ji}(h)
\]

The time interval between state transition from \( i \) to \( j \) is a random variable \( H_{ji} \), and we assume its distribution is exponential with given parameter \( \lambda_{ji} \).
HOMOGENEOUS MARKOV CHAIN

Therefore
\[ F_{j_i}^H (h) = P \left\{ H_{j_i} \leq h \right\} = 1 - e^{-\lambda_{j_i} h} \]
and
\[ p_{j_i}(h) = 1 - e^{-\lambda_{j_i} h} \]

We consider \( h = \Delta t \) to be a small increment of time and so it follows that:
\[ p_{j_i}(\Delta t) = 1 - e^{-\lambda_{j_i} \Delta t} \approx 1 - (1 - \lambda_{j_i} \Delta t) = \lambda_{j_i} \Delta t \]
We may write

\[ p_{ji}(\Delta t) \approx \lambda_{ji} \Delta t \]

Similarly, we have

\[ P \left\{ X(t + \Delta t) = i \mid X(t) = i \right\} = p_{ii}(\Delta t) \]

We also assume that \( p_{ii}(\Delta t) \) to be an approximately affine function of \( \Delta t \) with

\[ p_{ii}(\Delta t) \approx 1 - \lambda_i \Delta t \]
HOMOGENEOUS MARKOV CHAIN

- We can define the transition rates using

\[ \lambda_{ji} \triangleq \lim_{\Delta t \to 0} \frac{p_{ji} (\Delta t)}{\Delta t} \]

\[ \lambda_{ii} \triangleq \lim_{\Delta t \to 0} \frac{1 - p_{ii} (\Delta t)}{\Delta t} \]

- At each point in time, we must be in one of the \( m \) discrete states of the Markov chain
HOMOGENEOUS MARKOV CHAIN

Given that we are in state \( i \) at time \( t \), at \( t + \Delta t \) we are in either state \( i \) or some other state \( j \neq i \) and therefore

\[
p_{ii}(\Delta t) + \sum_{j \neq i} p_{ji}(\Delta t) = 1
\]

Therefore, we have the important relationship

\[
\lambda_i = \lim_{\Delta t \to 0} \frac{1 - p_{ii}(\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\sum_{j \neq i} p_{ji}(\Delta t)}{\Delta t} = \sum_{j \neq i} \lambda_{ji} \quad \forall i
\]
HOMOGENEOUS MARKOV CHAIN

- Transition diagram illustrates the time evolution

Diagram:

- Nodes: $i$, $j$, $i \neq j$
- Edges:
  - From $i$ to $j$ at time $t$
  - From $j$ to $i$ at time $t + \Delta t$
- Time axis: $t$, $t + \Delta t$
THE PROBABILITY TRANSITION MATRIX

We construct the probability transition matrix

\[ P(\Delta t) \], whose element \( j, i \) equals \( p_{ji}(\Delta t) \)

\[ P(\Delta t) \] has the characteristics:

\[ p_{ji}(\Delta t) \geq 0 \quad \forall i, \forall j \]

\[ \sum_j p_{ji}(\Delta t) = \sum_{j \neq i} p_{ji}(\Delta t) + p_{ii}(\Delta t) = 1 \quad \forall i \]

and so, the sum of every column of \( P(\Delta t) \) is 1
THE PROBABILITY TRANSITION MATRIX \( P(\Delta t) \)

- **Row** \( j \) represents the state from which the transition occurs.
- **Column** \( i \) represents the state to which the transition occurs.
- **Element** \( p_{ji}(\Delta t) \) is the probability of transitioning from state \( j \) to state \( i \) over the time interval \( \Delta t \).
- The sum of all the elements of each column \( i \) is 1.

\[ \text{from state} \rightarrow \text{column } i \rightarrow \text{sum of all the elements of each column } i \text{ is 1} \]

\[ \text{row } j \rightarrow \text{to state} \]
We define the *transition intensity matrix* for the discrete $m$-state space as

$$
\Lambda \triangleq \begin{bmatrix}
-\lambda_1 & \lambda_{12} & \cdots & \lambda_{1m} \\
\lambda_{21} & -\lambda_2 & \cdots & \lambda_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{m1} & \lambda_{m2} & \cdots & -\lambda_m
\end{bmatrix}
$$

It follows from the relations above that

$$
\Lambda = \lim_{\Delta t \to 0} \frac{P(\Delta t) - I}{\Delta t}
$$
APPLICATION OF $P(\Delta t)$

- Conditional probability establishes the relationship

$$P\{X(t+\Delta t) = i\} = \sum_{j=1}^{m} P\{X(t+\Delta t) = i \mid X(t) = j\} \cdot P\{X(t) = j\} \quad (*)$$

- Let

$$p_i(t) = P\{X(t) = i\}$$

and

$$p(t) = [p_1(t), p_2(t), \ldots, p_m(t)]^T$$
APPLICATION OF $P(\Delta t)$

Then, we may rewrite (**) as

$$p_i(t + \Delta t) = p_{ii}(\Delta t)p_i(t) + \sum_{j \neq i} p_{ij}(\Delta t)p_j(t)$$

$$i = 1, 2, ..., m$$

In vector form, we have

$$\underline{p}(t + \Delta t) = \underline{P}(\Delta t)\underline{p}(t)$$

(* *)

probability transition matrix
APPLICATION OF $P(\Delta t)$

- From (**) we obtain that

$$p(t + \Delta t) - p(t) = \left[ P(\Delta t) - I \right] p(t)$$

so that

$$\dot{p}(t) = \lim_{\Delta t \to 0} \frac{p(t + \Delta t) - p(t)}{\Delta t} = \lim_{\Delta t \to 0} \left[ \frac{P(\Delta t) - I}{\Delta t} \right] p(t) = \Lambda p(t)$$

- The solution of $\dot{p}(t) = \Lambda p(t)$ is

$$p(t) = e^{\Lambda t} p(0)$$
We determine the long-run probabilities $\underline{p}$ from the steady-state behavior of the system

$$\dot{\underline{p}}(t) = \Lambda \underline{p}(t)$$

with

$$\lim_{t \to \infty} \underline{p}(t) = \underline{p}$$

constant long-run probability vector
LONG–RUN PROBABILITIES

- The value of $p(t)$ as $t \to \infty$ corresponds to the equilibrium point

$$\lim_{t \to \infty} p(t) = 0 = \Lambda p$$

- We interpret the long-run probability $p$ as the fraction of time spent in the state $i$
LONG–RUN PROBABILITIES

☐ In the long-run, we have for \( j = 1, 2, \ldots, n \)

\[
\sum_{i \neq j} \lambda_{ji} p_i = \lambda_j p_j = \left( \sum_{i \neq j} \lambda_{ij} \right) p_j
\]

☐ Since at each \( t \), \( \sum_i p_i(t) = 1 \), this equality holds

also as \( t \to \infty \) and hence

\[
\sum_i p_i = 1
\]
THE SIMPLE TWO–STATE STOCHASTIC PROCESS

\[ \{X(t) : t \in \mathcal{T} \} \] is a stochastic process with two possible values
THE TWO–STATE MARKOV CHAIN

- The simple two-state stochastic process
  \[ \{X(t) : t \in \mathcal{T}\} \]
  may be viewed as a process of alternating up and down periods.

- We define the two r.v.s \( T_u \) and \( T_d \) for the uptime and downtime, respectively, of this process.

- The distributions of the uptime and downtime r.v.s are assumed to be exponential:
THE TWO–STATE MARKOV CHAIN

\[ F_{T_u}(t) = 1 - e^{-\lambda t} \quad \text{c.d.f. of the in-service time r.v.} \]
\[ F_{T_d}(t) = 1 - e^{-\mu t} \quad \text{c.d.f. of the repair time r.v.} \]

- The meaning of the parameters \( \lambda \) and \( \mu \) in the exponentially distributed r.v.’s \( T_u \) and \( T_d \) are:

\[
E\left\{ T_u \right\} = \frac{1}{\lambda}
\]
\[
E\left\{ T_d \right\} = \frac{1}{\mu}
\]
THE TWO–STATE MARKOV CHAIN

- In words, the mean up (down) time is the inverse of the rate of the in-service (repair) time \( r.v. \).

- Under these conditions, we have a two-state Markov chain with the up (down) state denoted as

1. \( 1(\theta) \) with transition probabilities

\[
P \left\{ X(t + \Delta t) = 0 \mid X(t) = 1 \right\} = p_{du}(\Delta t) \approx \lambda \Delta t
\]

\[
P \left\{ X(t + \Delta t) = 1 \mid X(t) = 0 \right\} = p_{ud}(\Delta t) \approx \mu \Delta t
\]
It follows that

\[ P \left\{ \bar{X}(t + \Delta t) = 1 \mid \bar{X}(t) = 1 \right\} = p_{uu}(\Delta t) \]
\[ \approx 1 - \lambda \Delta t \]

\[ P \left\{ \bar{X}(t + \Delta t) = 0 \mid \bar{X}(t) = 0 \right\} = p_{dd}(\Delta t) \]
\[ \approx 1 - \mu \Delta t \]
THE TWO–STATE MARKOV CHAIN

up

\( \mu \)

down

\( \lambda \)
THE TWO–STATE MARKOV CHAIN

We have the transition intensity matrix

\[ \Lambda = \begin{bmatrix} -\lambda_u & \lambda_{ud} \\ \lambda_{du} & -\lambda_d \end{bmatrix} = \begin{bmatrix} -\lambda & \mu \\ \lambda & -\mu \end{bmatrix} \]

The probability transition matrix is

\[ P(\Delta t) = \begin{bmatrix} 1 - \lambda \Delta t & \mu \Delta t \\ \lambda \Delta t & 1 - \mu \Delta t \end{bmatrix} \]
THE TWO–STATE MARKOV CHAIN

\[ \begin{align*}
1 - \lambda \Delta t \\
\mu \Delta t \\
\lambda \Delta t \\
1 - \mu \Delta t
\end{align*} \]
We compute the long-run probabilities from

\[ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\lambda & \mu \\ \lambda & -\mu \end{bmatrix} \begin{bmatrix} p_u \\ p_d \end{bmatrix} \]

or

\[ \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\lambda p_u + \mu p_d \\ \lambda p_u - \mu p_d \end{bmatrix} \]

We use the fact that

\[ p_u + p_d = 1 \]

to solve the two equations in the two unknowns.
Therefore

\[-\lambda p_u + \mu p_d = -\lambda (1 - p_d) + \mu p_d = -\lambda + (\lambda + \mu) p_d = 0\]

so that

The long-run probability that the system is in the down state

\[p_d = \frac{\lambda}{\lambda + \mu}\]

and

The long-run probability that the system is in the up state

\[p_u = \frac{\mu}{\lambda + \mu}\]
THE TWO–STATE MARKOV CHAIN

We recall that

\[ E \{ T_d \} = \frac{1}{\mu} \quad \text{and} \quad E \{ T_u \} = \frac{1}{\lambda} \]

so that we may rewrite

\[ p_d = \frac{1}{E \{ T_u \}} \cdot \frac{1}{E \{ T_d \}} = \frac{1}{E \{ T_u \} + E \{ T_d \}} = \frac{E \{ T_d \}}{E \{ T_u \} + E \{ T_d \}} \]
Similarly

\[ p_u = \frac{E\{T_u\}}{E\{T_d\} + E\{T_u\}} \]

We simplify the homogeneous two-state Markov chain to be a relatively simple transition system using the mean-time parameters of the two states.
MULTI–STATE REPRESENTATION

- We can represent the operation of a unit as a fractional or partial capacity unit; thus, the limit has three states:
  - full capacity $\Leftrightarrow u$
  - fractional capacity $\Leftrightarrow f$
  - zero capacity $\Leftrightarrow d$

- Each $\lambda_{ji}$ represents the intensity of transition from state $i$ to state $j$
3 – STATE SYSTEM REPRESENTATION

- Full capacity state
- Zero capacity state
- Fractional capacity state
We can also construct more complex systems using the simple two-state model for each unit.
STATE–SPACE REPRESENTATION

- Consider a system with three units each with capacity $c_i$ and intensities of transition $\lambda_i, \mu_i$,

  $i = 1, 2, 3$

- We construct the following 8-state Markov chain under the assumption that only one failure or one repair may occur during the time $\Delta t$.
THREE–UNIT SYSTEM MODEL
We associate frequency with the long-term behavior of the system.

We define the frequency of a state \( i \) to be the expected value of the rate

\[
\mathcal{F}_i \triangleq E \left\{ \begin{array}{l}
\text{number of} \\
\text{stays in} \\
\text{or}
\end{array} \right. \left\{ \begin{array}{l}
\text{arrivals into} \\
\text{or}
\end{array} \right. \left\{ \begin{array}{l}
\text{departures from} \\
\text{state } i / \text{unit time}
\end{array} \right. \} 
\]

with the computation performed over a long period.
The concept of frequency

- We denote by $F_{ji}$ the frequency of encountering state $j$ from state $i$.

- $F_{ji}$ is the expected value of the number of transitions from state $i$ to state $j$ per unit of time [rate of transition $i \rightarrow j$].
Consider the conceptual two-state process. We define $T_i \sim T_{-i}$ as the random duration of the system stay in state $i$ and $-i$.
FREQUENCY COMPUTATION

- Now, the cycle time is $T_i^c = T_i + T_{i-1}$

- Then, $E\left\{T_i^c\right\} = \text{mean duration of cycle time}$

- The frequency and duration are related through the relation

\[ f_i \cdot E\left\{T_i^c\right\} = 1 \]
Therefore,

\[ \mathcal{F}_i = \frac{1}{E\{T_i^c\}} = \frac{1}{E\{T_i\}} \cdot \frac{E\{T_i\}}{E\{T_i\}} = \frac{E\{T_i\}}{E\{T_i\} + E\{T_{\sim i}\}} \cdot \frac{1}{E\{T_i\}} \]

so that

\[ \mathcal{F}_i = p_i \frac{1}{E\{T_i\}} \]

and the duration of the system stay in state \( i \) is

\[ \mathcal{D}_i \triangleq E\{T_i\} \]
Consequently, we have the relationship that

\[ F_i = p_i \frac{1}{D_i} \]

or

\[ F_i D_i = p_i \]

In words, the long term fraction of time spent in state \( i \) is the product of the frequency of state \( i \) and the duration of stay in state \( i \).
COMPUTATION OF TRANSITION FREQUENCY

- We can evaluate $\mathcal{F}_{ji}$ along similar lines

- Recall that

$$\mathcal{F}_{ji} = E \left\{ \frac{\text{number of transitions from state } i \text{ to state } j}{\text{unit time}} \right\}$$

$$\mathcal{F}_{ji} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P \left\{ \{ X(t + \Delta t) = j \} \cap \{ X(t) = i \} \right\}$$
We have an important relationship that states that the frequency of transition from state $i$ to state $j$ is simply the product of the long term fraction of time spent in state $i$ and the transition rate from state $i$ to state $j$. 

\[
\begin{align*}
= & \lim_{\Delta t \to 0} \frac{1}{\Delta t} P \left\{ X(t + \Delta t) = j \mid X(t) = i \right\} P \left\{ X(t) = i \right\} \\
= & \lambda_{ji} p_i
\end{align*}
\]
FREQUENCY RELATIONSHIPS

Since

\[ F_i = E \left\{ \text{number of transitions into state } i \text{ / unit time} \right\} \]

\[ = \sum_{j \neq i} E \left\{ \text{number of transitions from state } j \text{ into state } i \text{ / unit time} \right\} \]

we have the relationship that

\[ F_i = \sum_{j \neq i} F_{ij} = \sum_{j \neq i} \lambda_{ij} p_j \]

We can show similarly that

\[ F_i = p_i \sum_{j \neq i} \lambda_{ji} \]
DURATION RELATIONSHIPS

- We use the relationships

\[ F_i = p_i \sum_{j \neq i} \lambda_{ji} \text{ and } F_i D_i = p_i \]

- to derive

\[ D_i = \frac{F_i D_i}{F_i} = \frac{p_i}{p_i \sum_{j \neq i} \lambda_{ji}} = \frac{1}{\sum_{j \neq i} \lambda_{ji}} \]

- In words, the duration in state \( i \) is the reciprocal of the sum of the transition intensities out of state \( i \)
THREE–STATE MODEL

The determination of $\mu_{ij}$ and $\lambda_{ij}$ is possible in terms of the parameters $P_i$, $\mathcal{F}_i$, and $\mathcal{D}_i$ for $i = 1, 2, 3$. 

\[
\begin{align*}
1 & \quad \mu_{12} \quad \mu_{13} \\
2 & \quad \lambda_{21} \quad \mu_{23} \\
3 & \quad \lambda_{31} \quad \lambda_{32}
\end{align*}
\]
GENERATION UNIT MODELING ISSUES

- Realistically, generation units are not as simple as described by the two-state r.v. $A_i$

- Each generation unit is
  - a complex system with multiple states
    - complete failure of unit [forced outage];
    - partial failures [continued operation at a reduced capacity – a so-called derated capacity state];
GENERATION UNIT MODELING ISSUES

- full capacity; or,

- removed from service for preventive and/or planned maintenance

- these states may further be impacted by time-dependent phenomena or, in certain cases, by energy limited operations
THE TWO–STATE MODEL

\[
\begin{align*}
\mathcal{D}_u &= \frac{1}{\lambda} \\
\mathcal{D}_d &= \frac{1}{\mu} \\
\text{unit unavailability} &= 1 - p = \frac{\mathcal{D}_d}{\mathcal{D}_u + \mathcal{D}_d} = \frac{\frac{1}{\mu}}{\frac{1}{\lambda} + \frac{1}{\mu}} = \frac{\frac{1}{\mathcal{D}_u}}{\frac{1}{\mathcal{D}_d}} = \frac{\mathcal{D}_d}{\mathcal{D}_u} = \frac{\lambda}{\mu + \lambda}
\end{align*}
\]
THE TWO–STATE MODEL

\[ D_u = \frac{1}{\lambda} \]

\[ D_d = \frac{1}{\mu} \]

unit availability = \( p = \frac{D_u}{D_u + D_d} = \frac{\mu}{\mu + \lambda} \)

1 – F.O.R.
NUMERICAL EXAMPLE OF TWO–STATE MODEL

- We consider a unit with the parameters

\[ p = 0.96 \]

and so

\[ 1 - p = 0.04 \]

- We derive that

\[ p = \frac{\mu}{\mu + \lambda} = 0.96 \]
NUMERICAL EXAMPLE OF TWO-STATE MODEL

\[ 0.96 \mu + 0.96 \lambda = \mu \]

\[ \mu = \frac{0.96}{0.04} \lambda \]

\[ = 24 \lambda \]

- The repair rate \( \mu \) is 24 times the failure rate \( \lambda \);

  equivalently, the duration of the \textit{up} state is 24 times that of the \textit{down} state.
Recall that

\[
FOR = \frac{FOH}{FOH + \text{available capacity hours}} = \frac{D_d}{D_d + D_u}
\]

Consequently, \( FOR \) and \( (1 - p) \) are identical quantities when computed over long periods of time.
THE TWO–STATE MODEL

- \((1 - p)\) is a good approximation of failure probability, even when preventive maintenance is considered, with the proviso that the maintenance state does not contribute to system failures; such an assumption is valid as long as the maintenance is performed on weekends or in low-load periods.
- We may account for partial outages by computing the equivalent FOH:
then, we replace \( FOR \) by \( EFOR \) with

\[
equivalent \ FOH = \begin{pmatrix}
\text{partial} \\
\text{outage} \\
\text{hours}
\end{pmatrix} \begin{pmatrix}
\text{fractional} \\
\text{capacity} \\
\text{reduction}
\end{pmatrix}
\]

\[
EFOR = \frac{FOH + \text{equivalent FOH}}{\text{available capacity hours} + FOH}
\]
We consider a system with four identical generating units with parameters $p$, $\lambda$, and $\mu$.

The state space and permitted transitions are:

- From state 0 (0 out) to state 1 (1 out) with rate $4\lambda$.
- From state 1 (1 out) to state 2 (2 out) with rate $3\lambda$.
- From state 2 (2 out) to state 3 (3 out) with rate $2\lambda$.
- From state 3 (3 out) to state 4 (4 out) with rate $\lambda$.
- From state 4 (4 out) to state 0 (0 out) with rate $4\mu$.
- From state 0 (0 out) to state 4 (4 out) with rate $2\mu$.
- From state 1 (1 out) to state 3 (3 out) with rate $3\mu$.
- From state 2 (2 out) to state 4 (4 out) with rate $4\mu$. 

Diagram:

- 0 (0 out) with rate $\mu$.
- 1 (1 out) with rate $2\mu$.
- 2 (2 out) with rate $3\mu$.
- 3 (3 out) with rate $4\mu$.
- 4 (4 out) with rate $4\mu$. 

We make use of the binomial distribution to analyze this example: for a system with $N$ units, let state $k$ denote the state corresponding to the failure of $k$ units; then,

$$p_k = \binom{N}{k} p^{N-k} (1-p)^k$$

and

$$F_k = p_k \left[ \lambda_{k^+} + \lambda_{k^-} \right],$$
the transition intensity from state $k$ to a state $k'$

with $k' > k \ (k' < k)$

We can show that

$$\lambda_{k^+} = (N - k) \lambda \quad \text{and} \quad \lambda_{k^-} = k \mu$$

For the simple example we use the values

<table>
<thead>
<tr>
<th>$c$</th>
<th>$p$</th>
<th>$\lambda$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50(MW)</td>
<td>0.96</td>
<td>0.4/year</td>
<td>9.6/year</td>
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## SIMPLE SYSTEM

<table>
<thead>
<tr>
<th>$k$</th>
<th>outage capacity</th>
<th>available capacity $x$</th>
<th>associated probability $P_k$</th>
<th>$P\left{ \sum_{i=i}^{4} \frac{A_i}{100} \leq x \right}$</th>
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