Problem 1

The equivalent load is defined by,

\[ L_k' = L_{k-1}' - A_k \]

(1)

\[ L_0' = L_0 \]

(2)

So to find the relation between \( L' \) and \( L \) will be necessary relate \( L_k' \) with \( L_k \). We start from definition (2) and adding \( c_k - c_k \),

\[ L_k' - L_{k-1}' + A_k c_k - c_k \]

(3)

\[ = L_{k-1}' + Z_k c_k \]

(4)

\[ = L_{k-2}' + Z_k c_{k-1} + Z_k c_k \]

(5)

\[ = L_{k-2}' + Z_k c_{k-1} + Z_k c_k \]

(6)

\[ = \ldots + \ldots + \ldots + \ldots \]

(7)

\[ = L_0' + \sum_{i=1}^{k} Z_i - \sum_{i=1}^{k} c_i \]

(8)

Using the fact that \( L_k = L_{k-1} + Z_k \), the relation of \( L_k' \) to \( L_k \) is,

\[ L_k' = L_k - C_k \]

(9)

So using the definition of \( L_k' \) we find,

\[ \mathcal{L}_k(x) = P(L_k' > x) = 1 - P(L_k' \leq x) = 1 - P(L_k \leq x + C_k) = P(L_k \leq x + C_k) = \mathcal{L}_k(x + C_k) \]

(10)

\[ \mathcal{L}_k(x) = \mathcal{L}_k(x + C_k) \]

(11)

Now we will derive convolution formula. The distribution of \( A_k \) is,

\[ f_{A_k}(x) = q_k \delta(x) + p_k \delta(x - c_k) \]

(12)

Then convolution formula,

\[ F_{L_k'}(x) = \int_{-\infty}^{\infty} F_{L_{k-1}'}(x + y) [q_k \delta(y) + p_k \delta(y - c_k)] dy \]

(13)

\[ = q_k F_{L_{k-1}'}(x) + p_k F_{L_{k-1}'}(x + c_k) \]

(14)

Then,

\[ 1 - F_{L_k'}(x) = 1 - q_k F_{L_{k-1}'}(x) - p_k F_{L_{k-1}'}(x + c_k) \]

(15)

\[ = q_k + p_k - q_k F_{L_{k-1}'}(x) - p_k F_{L_{k-1}'}(x + c_k) \]

(16)

\[ = q_k \left(1 - F_{L_{k-1}'}(x)\right) + p_k \left(1 - F_{L_{k-1}'}(x + c_k)\right) \]

(17)
So, finally,

\[ L'_k(x) = q_k L'_{k-1}(x) + p_k L'_k(x - c_k) \]  \hspace{1cm} (18)

**Problem 2**

Let's evaluate first \( E_k \). We know from lecture notes that in terms of \( L_k(x) \),

\[ E_k = p_k T \int_{C_{k-1}}^{C_k} L_{k-1}(x) dx \]  \hspace{1cm} (19)

From problem (1) we also know that,

\[ L'_k(x) = L_k(x + C_k) \rightarrow L_{k-1}(x) = L'_{k-1}(x - C_{k-1}) \]  \hspace{1cm} (20)

Put (20) into (19) we obtain,

\[ E_k = p_k T \int_{C_{k-1}}^{C_k} L'_{k-1}(x - C_{k-1}) dx \]  \hspace{1cm} (21)

We make a change of variables \( x - C_{k-1} = u \rightarrow dx = du \), and changing the limits of the integral, we obtain,

\[ E_k = p_k T \int_{0}^{C_k} L'_{k-1}(u) du \]  \hspace{1cm} (22)

In a similar way \( U_k \) in terms of \( L_k(x) \) is given by,

\[ U_k = T \int_{C_k}^{\infty} L_k(x) dx \]  \hspace{1cm} (23)

From (20) we have that \( L_k(x) = L'_k(x - C_k) \) so in (23) we obtain,

\[ U_k = T \int_{C_k}^{\infty} L'_k(x - C_k) dx \]  \hspace{1cm} (24)

Performing a change of variables \( x - C_k = u \rightarrow dx = du \), and changing the limits of the integral, we obtain,

\[ U_k = T \int_{0}^{\infty} L'_k(u) du \]  \hspace{1cm} (25)

**Problem 3**

We use the relation,

\[ L_{j-1}(x) = p_\alpha L_j(x) + q_\alpha L_j(x - c_i) \]  \hspace{1cm} (26)

in the expression that we have to check,

\[ L_j(x) = \frac{1}{p_\alpha} \left( \sum_{\nu=0}^{k-1} (-h)^\nu L_{j-1}(x - \nu c_i) \right) + (-h)^k \]  \hspace{1cm} (27)

where \((k - 1)c_i < x \leq kc_i\).
Then in the RHS we get,
\[
\frac{1}{p_\alpha} \left( \sum_{\nu=0}^{k-1} (-h)^\nu p_\alpha \mathcal{L}_I(x - \nu c_i) + (-h)^k q_\alpha (x - (\nu + 1) c_i) \right) + (-h)^k
\]  
(28)

Put in a explicit way important terms \( \nu = 0 \) in the first part of summation and \( \nu = k - 1 \) in the second part,
\[
\mathcal{L}_I(x) + \sum_{\nu=1}^{k-1} (-h)^\nu \mathcal{L}_I(x - \nu c_i) + \sum_{\nu=0}^{k-2} h(-h)^\nu \mathcal{L}_I(x - (\nu + 1) c_i) + \\
(h)(-h)^{k-1} \mathcal{L}_I(x - k c_i) + (-h)^k
\]  
(29)

(30)

Using the fact that \( \mathcal{L}_I(x - k c_i) = 1 \) in the set \( (k - 1)c_i < x \leq kc_i \), we can cancel the last two terms. We perform a change of variable in the second summation \( \nu + 1 = \mu \) we get,
\[
\mathcal{L}_I(x) + \sum_{\nu=1}^{k-1} (-h)^\nu \mathcal{L}_I(x - \nu c_i) - \sum_{\mu=1}^{k-1} (-h)^\mu \mathcal{L}_I(x - \mu c_i) = [\mathcal{L}_I(x)]
\]  
(31)

So we can see that the expression is correct, because the summations cancels out (dummy indexes) so the RHS is \( \mathcal{L}_I(x) \) like we expected.

**Problem 4**

From the relation \( L_{j-1} = I + Z \) the expression to show has to be (where we supposed that the first block of unit \( \alpha \) is loaded in position \( i - 1 \)):
\[
\mathcal{L}_{j-1}(x) = p_\alpha \mathcal{L}_I(x) + q_\alpha \mathcal{L}_I(x - c_i)
\]  
(32)

Let’s work in the RHS using ,
\[
\mathcal{L}_I(x) = \frac{1}{p_\alpha} \left( \sum_{\nu=0}^{k-1} (-h)^\nu \mathcal{L}_{j-1}(x - \nu c_i) \right) + (-h)^k
\]  
(33)

where \( (k - 1)c_i < x \leq kc_i \).

we obtain,
\[
\sum_{\nu=0}^{k-1} \mathcal{L}_{j-1}(x - \nu c_i) + p_\alpha (-h)^k + \frac{q_\alpha}{p_\alpha} \sum_{\nu=0}^{k-1} (-h)^\nu \mathcal{L}_{j-1}(x - (\nu + 1) c_i) + q_\alpha (-h)^k
\]  
(34)

Follow the same steps that problem (3), we expand explicitely some terms to see the cancelations,
\[
\mathcal{L}_{j-1}(x) + \sum_{\nu=1}^{k-1} \mathcal{L}_{j-1}(x - \nu c_i) - \sum_{\nu=0}^{k-2} (-h)^{\nu+1} \mathcal{L}_{j-1}(x - (\nu + 1) c_i) \\
- (-h)^k \mathcal{L}_{j-1}(x - k c_i) + (p_\alpha + q_\alpha)(-h)^k = [\mathcal{L}_{j-1}]
\]  
(35)

In this expression we use \( p_\alpha + q_\alpha = 1 \), \( \mathcal{L}_{j-1}(x - k c_i) = 1 \) in this set and the same change of variables in summations used in problem (3), to obtain that the RHS is \( \mathcal{L}_{j-1} \).
Problem 5

From lecture notes the cost $C_k$ as a function of the loaded capacity is,

$$C_k(\mu) = p_k T \left( w_k(c_k^{min})L_{k-1}[C_{k-1}] + \int_{C_{k-1}}^{C_{k-1}+\mu} w'_k[x-C_{k-1}]L_{k-1}dx \right)$$  \hspace{1cm} (36)

where $c_k \geq \mu \geq c_k^{min}$.

We also know that the expected energy in a range of energy $C_{k-1} < x < C_{k-1} + \mu$ is given by,

$$E_k(\mu) = p_k T \int_{C_{k-1}}^{C_{k-1}+\mu} \mathcal{L}_{k-1}(x)dx$$  \hspace{1cm} (37)

So if the marginal cost is constant and given by $\lambda_\beta$ then $w_k'(x) = \lambda_\beta$ when $x > c_k^{min}$ and $w_k'(x) = 0$ when $0 < x \leq c_k^{min}$. Using this information about $w_k'(x)$ in (36) we can write the integral that $\int_{C_{k-1}}^{C_{k-1}+\mu} - \int_{C_{k-1}}^{C_{k-1}+\mu+c_k^{min}}$, so the cost is given by,

$$C_k(\mu) = p_k T (w_k(c_k^{min})L_{k-1}[C_{k-1}] + \lambda_\beta[E_k(\mu) - E_k(c_k^{min})])$$  \hspace{1cm} (38)

Because in this case we have at least one block loaded, there is no constrain due to $c_k^{min}$ because this was considered in the charge of the first block, with this in mind is clear that the extra cost of aditional blocks will be only the term $\lambda_\beta E_k(\mu)$. In other words the integral

$$\int_{C_{k-1}}^{C_{k-1}+\mu} w_k'[x-C_{k-1}]L_{k-1}dx$$  \hspace{1cm} (39)

is split in two integrals, one for the first blocks and the additional,

$$\int_{C_{k-1}}^{C_{k-1}+\mu} w_k'[x-C_{k-1}]L_{k-1}dx = \int_{C_{k-1}}^{C_{k-1}+\mu_{block1}} w_k'[x-C_{k-1}]L_{k-1}dx + \int_{C_{k-1}+\mu_{block1}}^{C_{k-1}+\mu_{additional}} w_k'[x-C_{k-1}]L_{k-1}dx$$  \hspace{1cm} (40)

the first integral gives an expression like (38) evaluate in $\mu = \mu_{block1}$. Because in all the interval $C_{k-1} + \mu_{block1} < x < C_{k-1} + \mu_{additional}$ the unitary cost is $w_k'(x) = \lambda_\beta$ we don’t have the term $c_k^{min}$ and the extra cost of this block will be $\lambda_\beta E_k$.

Problem 6

The three state unit will have the distribution function,

$$f_A(x) = q_1 \delta(x) + r_1 \delta(x-d_1) + s_1 \delta(x-c_1)$$  \hspace{1cm} (41)

We use strategy similar to problem (1) of Problem Set 1, we can find the convolution formula in this case,

$$M = \sum_{j=1}^{i-1} A_j$$  \hspace{1cm} (42)

and

$$Y = A_i$$  \hspace{1cm} (43)

, then

$$P(\sum_{i=1}^{i} A_i \leq x) = F_{M+Y}(x) = \int_{-\infty}^{x} F_{M}(x-m) f_Y(m) dm$$  \hspace{1cm} (44)
\[
P(\sum_{i=1}^{i} A_i \leq x) = q_i P(M \leq x) + r_i P(M \leq x - d_i) + s_i P(M \leq x - c_i)
\]  

(45)

Now we can do a similar analysis to two level unit, in the context of production costing. Let’s define the variable \( Z_i = c_i - A_i \), this has the distribution given by,

\[
f_Z_i = s_i \delta(x) + r_i \delta(x - (c_i - d_i)) + q_i \delta(x - c_i)
\]  

(46)

So to evaluate \( L_i = L_{i-1} + Z_i \) we use convolution formula,

\[
F_{L_i} = \int F_{L_{i-1}}(x - y) f_Z_i(y) dy
\]  

(47)

Using (46) in the last expression we obtain,

\[
1 \quad F_{L_i} = s_i F_{L_{i-1}}(x) + r_i F_{L_{i-1}}(x - (c_i - d_i)) + q_i F_{L_{i-1}}(x - c_i)
\]  

(48)

\[
\mathcal{L}_i(x) = s_i \left[ 1 - F_{L_{i-1}}(x) \right] + r_i \left[ 1 - F_{L_{i-1}}(x - (c_i - d_i)) \right] + q_i \left[ F_{L_{i-1}}(x - c_i) \right]
\]  

(49)

So the expression in this case becomes,

\[
\mathcal{L}_i(x) = s_i \mathcal{L}_{i-1}(x) + q_i \mathcal{L}_{i-1}(x - c_i) + r_i \mathcal{L}_{i-1}(x - (c_i - d_i))
\]  

(50)

For the energy, we take the relation given in lecture notes,

\[
E_i = U_{i-1} - U_i
\]  

(51)

\[
= T \int_{C_{i-1}}^{\infty} \mathcal{L}_{i-1}(x) dx - T \int_{C_i}^{\infty} \mathcal{L}_i(x) dx
\]  

(52)

Using relation (50) into (52) we obtain,

\[
\frac{E_i}{T} = \int_{C_{i-1}}^{\infty} \mathcal{L}_{i-1}(x) dx - s_i \int_{C_i}^{\infty} \mathcal{L}_{i-1}(x) dx - r_i \int_{C_{i-1}}^{\infty} \mathcal{L}_{i-1}(x - (c_i - d_i)) dx - q_i \int_{C_i}^{\infty} \mathcal{L}_{i-1}(x - c_i) dx
\]  

(53)

\[
= (s_i + r_i + q_i) \int_{C_{i-1}}^{\infty} \mathcal{L}_{i-1}(x) dx - s_i \int_{C_i}^{\infty} \mathcal{L}_{i-1}(x) dx - r_i \int_{C_{i-1}+d_i}^{\infty} \mathcal{L}_{i-1}(x) dx - q_i \int_{C_i}^{\infty} \mathcal{L}_{i-1}(x) dx
\]  

(54)

Where in the last line we made a change of variables in the integral. Simplifying a little bit we obtain,

\[
\frac{E_i}{T} = r_i \int_{C_{i-1}}^{C_{i-1}+d_i} \mathcal{L}_{i-1}(x) dx + s_i \int_{C_{i-1}+c_i}^{C_{i-1}+d_i} \mathcal{L}_{i-1}(x) dx
\]  

(55)

And split the last integral we obtain the expression,

\[
\frac{E_i}{T} = r_i \int_{C_{i-1}}^{C_{i-1}+d_i} \mathcal{L}_{i-1}(x) dx + s_i \int_{C_{i-1}}^{C_{i-1}+d_i} \mathcal{L}_{i-1}(x) dx + s_i \int_{C_{i-1}+d_i}^{C_{i-1}+2d_i} \mathcal{L}_{i-1}(x) dx
\]  

(56)