University of Illinois **Fall 2018**

ECE 586GT: Problem Set 5: Problems and Solutions
Games with incomplete information, multistage games, VCG mechanisims

Due: Thursday, November 15, at beginning of class
Reading: Course notes, Section 4.3 and Chapters 5 & 6

1. **[Supporting a public good with incomplete information]**
Consider n players, any of whom could perform a job that would benefit all the players. The players don't necessarily give the job high enough priority to coordinate about it. For example, the players could be participating in some venture and the job is to make preparations to ensure safety in case of some unanticipated event. Suppose player i selects $a_i \in \{0,1\}$, where $a_i = 1$ indicates that the player does the job. Suppose each player derives value 1 if at least one player does the job. Suppose θ_i is the type of player i, which represents the cost for player i to do the job. So the payoff of player i for a given action profile $a = (a_1, \ldots, a_n)$ and θ_i is $u_i(a, \theta_i) = -\theta_i a_i + \mathbf{1}_{\{\sum_{i\in[n]} a_i \geq 1\}}$. Player i can use the value of θ_i to select a_i; the other players don't see θ_i. Suppose the θ_i's are mutually independent and uniformly distributed over $[0,1]$ for $i \in [n]$.

   (a) Identify all the Bayes-Nash equilibria in pure strategies for $n = 2$. (A pure strategy for player i is a mapping $s_i : [0,1] \to \{0,1\}$, used to map θ_i to an action a_i.)
   Solution: Fix a strategy profile $(s_i(\cdot))_{i\in[n]}$ and consider a player i. Let $r_j = \mathbb{P}\{s_j(\theta_j) = 1\} = \int_0^1 s_j(\theta_j) d\theta_j$. The expected payoff of player i given θ_i for action 0 or 1 is given by

$$\mathbb{E}[u_i | a_i = 0, \theta_i] = -0 \; + \left[1 - \prod_{j:j\neq i}(1 - r_j)\right]$$
$$\mathbb{E}[u_i | a_i = 1, \theta_i] = -\theta_i + 1.$$

   Thus, the best response $s_i(\cdot)$ to the strategies of the other players is given by

$$s_i(\theta_i) = \mathbf{1}_{\{\theta_i \leq \prod_{j:j\neq i}(1-r_j)\}}.$$

   Thus, for any Bayes-Nash equilibrium, the strategy functions are threshold type, with the threshold for player j being r_j for $j \in [n]$. In other words $s_j(\theta_j) = \mathbf{1}_{\{\theta_j \leq r_j\}}$, and, moreover, the thresholds satisfy $r_i = \prod_{j:j\neq i}(1 - r_j)$ for all $i \in [n]$.
   For the special case $n = 2$ the equations for the thresholds both reduce to $r_1 + r_2 = 1$. Thus, in case $n = 2$, there is a one-parameter family of Bayes-Nash equilibia, given by $(s_1(\theta_1) = \mathbf{1}_{\{\theta_1 \leq r\}}, s_2(\theta_2) = \mathbf{1}_{\{\theta_2 \leq 1-r\}})$, for $0 \leq r \leq 1$.

   (b) Identify all the Bayes-Nash equilibria in pure strategies for $n = 3$. (This problem is similar for any $n \geq 3$, but for simplicity you may stick to $n = 3$.)
   Solution: For $n = 3$ the thresholds satisfy the equations

$$r_1 = (1 - r_2)(1 - r_3)$$
$$r_2 = (1 - r_1)(1 - r_3)$$
$$r_3 = (1 - r_1)(1 - r_2).$$

Multiplying the i^{th} equation by $1-r_i$ we get $r_i(1-r_i) = \prod_j(1-r_j)$ for all i. Thus, there is an $r \in [0,1]$ so that $r_i(1-r_i) = r(1-r)$ for all i, or, equivalently, $r_i \in \{r, 1-r\}$ for all i. We have two cases.
Case 1: $r_1 = r_2 = r_3$ In this case we can assume $r_i = r$ for all i, where r is the unique solution to $r = (1-r)^2$, or $r = \frac{3-\sqrt{5}}{2} \approx 0.382$.
Case 2: After possibly permuting the players and/or changing r to $1-r$, $r_1 = r$ and $r_2 = r_3 = 1-r$. In this case the equations for the thresholds reduce to $r = r^2$ and $1-r = r(1-r)$, which are both satisfied if and only if $r = 0$.

In summary, the Bayes-Nash equilibria for $n = 3$ consist of the three strategy profiles $(1,0,0), (0,1,0), (0,0,1)$, for which the strategy functions of the players are each constant, and the symmetric strategy profile such that $s_i(\theta_i) = \mathbf{1}_{\{\theta_i \leq r\}}$ for the threshold $r = \frac{3-\sqrt{5}}{2} \approx 0.382$.

2. **[Repeated zero sum games]**
Consider infinite repeated play of a finite, two-player zero sum game with some discount factor δ. What conclusion can be drawn from Nash's realization theorem for such a game?

Solution: (In this answer we assume the stage game has an objective function ℓ and player 1 seeks to minimize it and player 2 seeks to maximize it.) By the theory of such games, the minmax value of player 1 is equal to the maxmin payoff of player 2, so there is no other vector of payoffs that is realizable other than the payoffs for the Nash equilibrium.

3. **[Wide world of subgame perfect equilibria for repeated games]**
Consider infinite repeated play of the prisoners dilema game, with payoff matrix

	C	D
C	1,1	-1,2
D	2,-1	0,0

Let $n \geq 1$. Find $\bar{\delta}(n) \in (0,1)$ so that for any $\delta \in [\bar{\delta}, 1)$ and any script of plays of length n, there is a subgame perfect equilibrium for the infinite repeated game with discount factor δ such that, when used, the players both follow the given script for the first n plays. (You don't need to find the minimum possible value.)

Solution: Extend the script to infinite length by using action profile (C, C) in all stages after the n^{th} stage. Let the strategy of player i be to follow the script in each stage as long as both players followed the script in all previous stages. If any player does not follow the script in some stage then both players use action D in all subsequent stages. Let's examine when this pair of strategies is a subgame perfect equilibrium for the infinite repeated game with discount factor δ for some $\delta \in (0,1)$.

By the one-step deviation principle for infinite-horizon repeated games, it suffices to consider perturbations to the strategy profile such that one player in one state takes a different action. If no such change leads to an increase in expected payoff for the game starting in that stage, the game is subgame perfect.

If there is a scenario so that such an action could make a positive difference, then there is a scenario such that the change happens in the first stage and for player 1, because starting in the first phase leaves the greatest number of stages from the original script, and the game is symmetric in the players. Also, it is a worst case for the script to be (C, D) in stages 2 through n, because that gives player 1 the minimum payoff of -1 in stages 2 through n. Also, it is worst case for the scripted action of player 1 to be C in the first stage. We consider both cases C or D for the scripted action of player 2 in the first stage. We thus conclude that the strategy profile is subgame perfect if the payoff sequence $(\underbrace{1,-1,-1,\ldots,-1}_{n},1,1,\ldots)$ has total payoff at least as large as $(2,0,0,\ldots)$, (covers the case player 2 starts with C, in which case the possible payoffs of player 1 are 1 if follow the script and play C, or 2 if deviate and play D) and if $(\underbrace{-1,\ldots,-1}_{n},1,1,\ldots)$ has total payoff at least as large as $(0,0,0,\ldots)$ (covers the case player 2 starts with D, in which case the possible payoffs of player 1 are -1 if follow the script and play C, or 0 if deviate and play D)). The first of these conditions reduces to the second, so we can drop the first condition. Since stage k is weighted by $(1-\delta)\delta^{k-1}$ for $k \geq 1$, the second condition becomes

$$(1-\delta)\left(-\frac{1-\delta^n}{1-\delta}+\frac{\delta^n}{1-\delta}\right) \geq 0$$

or $\delta^n \geq 1/2$. So if $\bar{\delta} = (1/2)^{1/n}$, then the strategy profile is subgame perfect if $1 > \delta \geq \bar{\delta}$.

4. **[Repeated play for the Bertrand equilibrium problem]**
Suppose n players, for some $n \geq 2$, represent firms that can each produce a common divisible good at a cost c per unit of good. Suppose the action of each player is to declare a price p_i per unit of good. Suppose there is an aggregate demand of consumers such that if the lowest price offered by any player is p_{min} then the consumers purchase a total quantity $(a-p_{\min})_+$ of good, where a is a constant with $a > c$, and they purchase an equal amount from each player offering the minimum price. The game is among the players offering prices; the consumers are not considered to be part of the game. As shown in ECE 586GT problem set 1, problem 4(a), 2017, p is a Nash equilibrium if and only if $p_{\min} = c$ and $p_i = c$ for at least two players. The resulting payoff vector is the all zero vector. Basically, the competition among the players drives the sum of payoffs to zero at Nash equilibrium.

   (a) Let's see if positive payoffs can be sustained in the infinite repeated game with the Bertrand based game as a stage game, with weight $(1-\delta)\delta^{t-1}$ on stage t. To begin, fix $\bar{p}$ with $c < \bar{p} < \frac{a+c}{2}$, and consider the trigger strategy based on the script that every player bids price $\bar{p}$ in every stage. Players follow the script, with switching to playing c forever if any player deviated from the script in an earlier stage. Each player will receive payoff $(a-\bar{p})(\bar{p}-c)/n$ per stage. Find the minimum δ so that such strategy profile is subgame perfect.

   Solution: By the one-step deviation principle for infinite-horizon repeated games, it suffices to consider one player deviating from the trigger strategy in one stage. By symmetry and time invariance we can consider without loss of generality a change for player 1 in the first stage. The player should select an action to maximize the payoff in the first stage, because the payoffs in all subsequent stages will be zero. Thus, the player should bid $\bar{p}-\epsilon$ for a very small value of ϵ, resulting in payoff $(a-\bar{p}+\epsilon)(\bar{p}-c-\epsilon)$, which is smaller, but can be made arbitrarily close to, $(a-\bar{p})(\bar{p}-c)$. Thus, player 1 has no incentive to deviate if and only if $(1-\delta)(a-\bar{p})(\bar{p}-c) \leq (a-\bar{p})(\bar{p}-c)/n$, or equivalently, if and only if $1-\delta \leq \frac{1}{n}$, or $\delta \geq \frac{n-1}{n}$. Interestingly, this condition doesn't depend on $\bar{p}$. The payoffs would be maximized by taking $\bar{p} = \frac{a+c}{2}$.

   (b) Identify the set of payoff vectors v in the Nash realization region for the infinite repeated game with stage game based on the Bertrand selling mechanism.

   Solution: Let $\bar{p} = \frac{a+c}{2}$ and $P = \frac{(a-c)^2}{4}$. Note that P is the maximum sum of payoffs. If player i bids $\bar{p}$ and all other players make larger bids, then player gets payoff P and all other players get 0 payoff. Therefore, the vectors $(P, 0, 0, \ldots, 0), (0, P, 0, \ldots, 0), \ldots, (0, \ldots, 0, P)$ as well as $(0, 0, \ldots, 0)$, are feasible. The convex hull is $\{v \in \mathbb{R}^n : v \geq 0, \sum_{i\in[n]} v_i \leq P\}$, and the Nash realization region is therefore $\{v \in (0, +\infty)^n : \sum_{i\in[n]} v_i \leq P\}$. (Since the players revert to a Nash equilibrium, the points in the Nash realization region are realizable by subgame perfect equilibria as well, by Friedman's theorem.)

5. **[Locating a center by VCG mechanism.]**
Suppose there are n players indexed by $[n]$, such that each player i has a location $x_i \in \mathcal{C}$ for some set $\mathcal{C}$. The problem is to find a mechanism to select a center (i.e. central location), $y \in \mathcal{C}$, and specify payments the players need to make if they choose to influence the choice of y through a bidding process. Suppose each player i has a type, $\theta_i > 0$, that represents how important it is for player i to be near y. Let ℓ be a function such that $\ell : C \times C \to \mathbb{R}_+$. The locations x_i are known to the decision mechanism, and the variable θ_i is private information of player i for each i. Suppose the total loss of player i is given by $L_i = \theta_i \ell(x_i, y) + m_i$, where m_i is the amount of money charged to player i by the mechanism.

(a) Describe use of a VCG mechanism to seclect $y \in \mathcal{C}$ and determine the payments $(m_i)_{i\in[n]}$, based on bids and the known locations of the players. To be definite select the payment rule so that the minimum payment possible for any given player is 0. (Hint: Since the loss function of player i has the form $y \mapsto \theta_i \ell(x_i, y)$ and x_i is known to the mechanism, the family of loss functions is one dimensional, so the bids can be one dimensional. Here the loss functions play the usual role of valuation functions, but with players seeking to minimize losses instead of maximizing valuations.)

Solution: Following the hint, let the space of bids be $S_i = (0, \infty)$ for each i, and let $\hat{\theta}_i$ denote the bid of player i. The interpretation is that $\hat{\theta}_i$ represents the valuation function $y \mapsto \hat{\theta}_i \ell(x_i, y)$. Truthful bidding means $\hat{\theta}_i = \theta_i$. The VCG allocation mechanism is $y = g(\hat{\theta}, x)$ where

$$y^* \in \arg\min_{y' \in \mathcal{C}} \sum_{i\in[n]} \hat{\theta}_i \ell(x_i, y')$$

and payments $(m_i)_{i\in[n]}$ such that

$$m_i = \sum_{j:j\neq i} \hat{\theta}_j \ell(x_j, y^*) - \sum_{j:j\neq i} \hat{\theta}_j \ell(x_j, y_i^*),$$

where

$$y_i^* \in \arg\min_{y_i \in \mathcal{C}} \sum_{j:j\neq i} \hat{\theta}_j \ell(x_j, y_i).$$

Note that y_i^* is the location of the center if player i were ignored, based on the reported types of the other players.

(b) Simplify your expressions in case $\mathcal{C} = \mathbb{R}^d$ for some $d \geq 1$, and ℓ is squared Euclidean distance: $\ell(x, y) = \|x - y\|^2$. (Hint: Express the payment m_i using of y_i^*.)

Solution: The allocation y^* is the weighted center of mass of the locations:

$$y^* = \arg\min_{y' \in \mathbb{R}^d} \sum_{i\in[n]} \hat{\theta}_i \|x_i - y'\|^2 = \frac{\sum_{i\in[n]} \hat{\theta}_i x_i}{\sum_{i\in[n]} \hat{\theta}_i},$$

because the minimum mean squared error estimate of a random vector is the mean of the vector. Similarly, for each i,

$$\sum_{j:j\neq i} \theta_j (y_i^* - x_j) = 0. \tag{1}$$

Using (1), the payment rule can be expressed as:

$$\begin{aligned} m_i &= \sum_{j:j\neq i} \hat{\theta}_j \left(\|x_j - y^*\|^2 - \|x_j - y_i^*\|^2 \right) \\ &= \sum_{j:j\neq i} \hat{\theta}_j \left(-2\langle x_j, y^* - y_i^* \rangle + \|y^*\|^2 - \|y_i^*\|^2 \right) \\ &= \sum_{j:j\neq i} \hat{\theta}_j \left(-2\langle y_i^*, y^* - y_i^* \rangle + \|y^*\|^2 - \|y_i^*\|^2 \right) \\ &= \|y_i^* - y^*\|^2 \left(\sum_{j:j\neq i} \hat{\theta}_j \right) \end{aligned}$$

In words, the payment of player i is based on the squared distance between the weighted centers of mass when player i is either accounted for or excluded, times the sum of the bids of the other players.

(c) Find numerical values of y^* and the payments for part (b) in case $d = 1$, $n = 4$, $\theta_i = x_i = i$ for $i \in [n]$. (Hint: In the end you should find that the payment of player 3 happens to be zero.)

Solution: The allocation, other weighted centers of mass, and payments are given by

$$\begin{aligned} y^* &= \tfrac{1^2+2^2+3^2+4^2}{10} = 3 & & \\ y_1^* &= \tfrac{2^2+3^2+4^2}{9} = \tfrac{29}{9} & m_1 &= 9(2/9)^2 = 4/9 = 0.444 \\ y_2^* &= \tfrac{1^2+3^2+4^2}{8} = \tfrac{26}{8} & m_2 &= 8(2/8)^2 = 4/8 = 0.5 \\ y_3^* &= \tfrac{1^2+2^2+4^2}{7} = 3 & m_3 &= 0 \\ y_4^* &= \tfrac{1^2+2^2+3^2}{6} = \tfrac{14}{6} & m_4 &= 6(4/6)^2 = 16/6 = 2.666 \end{aligned}$$

The payment of player 4 is the largest because player 4 had a relatively large bid and the location of player 4 is at one end of the constellation, causing a large pull on the center of mass. Player 3 is lucky to be located at the weighted center of mass of the other three players and hence pays zero.

6. **[VCG applied to combinatorial auction]**
Suppose a VCG mechanism is applied to sell the objects in $\mathcal{O} = \{a, b, c\}$ to three bidders, who have submitted the following bids:

$$\begin{array}{lll} v_1\{a\} = 12, & v_1\{a,b\} = 14 & \\ v_2\{b,c\} = 6, & v_2\{a,b,c\} = 12 & \\ v_3\{b\} = 2, & v_3\{c\} = 5, & v_3\{b,c\} = 6. \end{array}$$

Assume the value of any bundle to any bidder, if not listed above, is zero. Also, assume that for any bidder, at most one of the bidder's bids may be used. In particular, by bidding such that $v_3\{b,c\} < v_3\{b\} + v_3\{c\}$, bidder 3 communicates that b and c are partial substitutes for the bidder. Suppose also that the seller has a reserve price, or value, of 1, for each item that is not sold. Determine the assignment of objects to bidders and the payments of the bidders, for the VCG mechanism under truthful bidding. Use the individually rational version of payments.

Solution: The VCG allocation is unique, and is given by $A^* = (\{a,b\}, \emptyset, \{c\})$, with total value 19. To find m_1, note that if bidder 1 were not present, the maximum welfare assignment would be $(-, \{a,b,c\}, \emptyset)$ with total value 12, whereas bidders 2 and 3 receive total value 5 under A^*, so $m_1 = 12 - 5 = 7$. To find m_3, note that if bidder 3 were not present, the maximum welfare assignment would be $(\{a\}, \{b,c\}, \emptyset, -)$, for a total value 12+6 $= 18$, whereas bidders 1 and 2 receive total value 14 under A^*, so $m_3 = 18 - 14 = 4$. Also, $m_2 = 0$ because bidder 2 is assigned no objects under A^*. So the payment profile is $m = (7, 0, 4)$.

Another way to identify m_1 is to replace player 1 by a single-minded version as shown (see Example 6.6 of the notes):

$$\begin{array}{lll} v_1'\{a,b\} = m_1 & & \\ v_2\{b,c\} = 6, & v_2\{a,b,c\} = 12 & \\ v_3\{b\} = 2, & v_3\{c\} = 5, & v_3\{b,c\} = 6. \end{array}$$

Then note that the smallest value of m_1 such that A^* continues to be the maximum welfare assignment is $m_1 = 7$.