

ECE 586GT: Problem Set 1: Problems and Solutions
Analysis of static games

Due: Tuesday, Sept. 11, at beginning of class

Reading: Course notes, Sections 1.1 - 1.4

1. [A random zero sum game]

Consider a two-player zero-sum game corresponding to the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The players select an element of the matrix in the following way: player 1 selects a row and player 2 selects a column. The selected element is the payoff of player 2 and the negative of the payoff of player 1. Thus, player 1 seeks to minimize the element selected and player 2 seeks to maximize it. Suppose that A, B, C, D are independent and identically distributed continuous-type random variables. Suppose both players know the values of the random variables before selecting their actions. (Hint: The event two or more of the random variables are equal has probability zero, so assume without loss of generality that the random variables take different values. By symmetry, the probability any k of the random variables have a particular order is $1/k!$. For example, $\mathbb{P}\{A < B < C\} = 1/6$.)

- (a) Find the probability there is more than one pure-strategy Nash equilibrium (NE).

Solution: There are four possible pure strategy profiles, which we refer to by their payoffs for player 2. Profile A is an NE if and only if $B < A < C$. So if A is an NE, neither B or C can be an NE. Similarly, D is an NE if and only if $C > D > B$. Since $\{B < C\} \cap \{B > C\} = \emptyset$, A and D cannot both be NEs. Thus, the probability there is more than one pure-strategy NE is zero.

- (b) Find the probability there is at least one pure-strategy NE.

Solution: Profile A is an NE if and only if $B < A < C$, which has probability $1/6$. By symmetry, any of the four profiles has probability $1/6$ to be an NE, and the probability two or more of them are NEs is zero. Thus, the probability there is at least one NE is $4/6 = 2/3$.

- (c) Find the probability player 1 has a dominant strategy.

Solution: The first row is a dominant strategy if and only if $\{A < B\} \cap \{C < D\}$, which has probability $1/4$. Similarly, the second row is a dominant strategy with probability $1/4$. Thus, the probability player 1 has a dominant strategy is $1/4 + 1/4$, or $1/2$.

- (d) Find the probability both players have a dominant strategy.

Solution: If both players have a dominant strategy, then both players using their dominant strategy is a strategy profile. The event that strategy profile A represents a pair of dominant strategies is the event $A < C$, $B < D$, $A > B$, and $C > D$. In other words, it is the event that B is the smallest and C is the largest of the four random variables, or it is the union of the two events $B < A < D < C$ and $B < D < A < C$, which has probability $2/4! = 1/12$. Hence, the probability both players have a dominant strategy is $4/12 = 1/3$. (Comparing to part (b), we can conclude that given there exists an NE, the conditional probability both players have dominant strategies is $1/2$.)

2. [Territory control game on a graph]

An application of the game in this problem could be for each of a set of players to decide where to place their fast food restaurant, under the assumption any customers will travel to the nearest fast food restaurant. Suppose $n \geq 2$ and $G = (V, E)$ is an undirected graph with vertex set V and edge set E . Consider the n player game such that the action of each player is to select a vertex in V . Two or more players can select the same vertex. Given the set of vertices selected by the players, each vertex v in the graph assigns total payoff one among the players, divided equally among those players who selected vertices the closest to the vertex v , where the distance between two vertices is the minimum number of edges in a path from one vertex to the other. For example, if there are no ties, the vertex v assigns payoff one to the player with selected vertex closest to v . The total payoff of a player is the sum over all vertices of the payoff assigned to that player. Thus, the sum of the payoffs of all players is $|V|$ for any choices of the players.

- (a) Does there always exist at least one Nash equilibrium in mixed strategies?

Solution: Yes, by Nash's theorem for finite games.

- (b) Consider the line graph G with $V = \{1, 2, \dots, 100\}$ and $E = \{[i, i + 1] : 1 \leq i \leq 99\}$. Suppose there are two players ($n = 2$). Find the set of all pure strategy Nash equilibria.

Solution: Suppose (s_1, s_2) is a NE. It is necessary that $|s_1 - s_2| \leq 1$, or else either player i could increase his/her payoff by moving s_i closer to s_{-i} . Given that, it is also necessary that $\{s_1, s_2\} \in \{50, 51\}$, or else one of the players i selecting an action furthest from $\{50, 51\}$ could get a larger payoff by moving s_i one step closer to $\{50, 51\}$. That leaves us with the following possible strategy profiles: $(50, 50)$, $(50, 51)$, $(51, 50)$, $(51, 51)$, and it is easily checked that all four of these strategy profiles are NE.

- (c) Is there a dominant strategy for a player in the game of part (b)?

Solution: No. For example, if player 1 selects s_1 with $s_1 \leq 49$, the unique best response of player 2 is to select $s_2 = s_1 + 1$. Thus, there is no response for player 2 that is optimal for any choice made by player 1.

- (d) Repeat part (b) for three players, $n = 3$.

Solution: We show by argument by contradiction that there is no NE in pure strategies for $n = 3$. For the sake of argument by contradiction, suppose (s_1, s_2, s_3) is a NE. Reorder the strategies if necessary so that $s_1 \leq s_2 \leq s_3$. It is necessary that $s_2 - s_1 \leq 1$, or else player 1 could get a larger payoff by increasing s_1 by one. It is similarly necessary that $s_3 - s_2 \leq 1$. It is impossible for $s_1 = s_2 = s_3$, because in that case any one player could increase his/her payoff from $100/3$ to at least 50 by changing his/her action by one. It is also impossible for s_1, s_2, s_3 to be consecutive integers, because player 2 would be able to increase his/her payoff from 1 to at least 49. The remaining possibilities are when two of the actions are equal and the third is different by one. So consider the action profile $(k, k, k + 1)$ for some integer k with $1 \leq k \leq 99$. Cases $k = 1$ and $k = 99$ can be easily eliminated separately, so we can restrict attention to $2 \leq k \leq 98$. The payoffs for the three players are $k/2, k/2, 100 - k$ respectively. If player 1 changed strategy to $k + 2$, his/her payoff would change to $100 - k - 1$, so it must be that $100 - k - 1 \leq k/2$ or $k \geq 66$. If player 3 changed strategy to $k - 1$, his/her payoff would change to $k - 1$, implying that $k - 1 \leq 100 - k$ or $2k \leq 101$ or $k \leq 50$. Thus, if order for $(k, k, k + 1)$ to be an NE, it is necessary that $k \geq 66$ and $k \leq 50$, which is impossible. There are no remaining possibilities for (s_1, s_2, s_3) to be an NE; the proof by contradiction is complete.

3. **[Possession is nine-tenths of the law]**

Consider the normal form game $G = (I, (S_i), (u_i))$ with $I = \{1, \dots, n\} = [n]$, $S_i = [0, 1]$, and $u_i(s_i) = \begin{cases} s_i & \text{if } s_1 + \dots + s_n \leq 1 \\ 0 & \text{else} \end{cases}$. For example, suppose there is a pie, and every player declares what fraction of the pie he/she will take. Players get what they declare if the sum of the fractions is less than or equal to one. Else none get any pie.

- (a) Find all dominant strategies for a given player. Justify your answer.

Solution: There are none. For a given player i and any $x \in [0, 1]$, it is possible that the sum of the actions of other players is x . Then the unique best response of player i is $s_i = 1 - x$. Therefore, there is no single action of player i that is always a best response.

- (b) Find all Nash equilibria in pure strategies. (Include them all. Even ones such that all players get zero payoff.)

Solution: Let $s = (s_1, \dots, s_n)$ be a strategy profile. If $\sum_i s_i < 1$ then s is not a NE, because any player could increase his/her payoff by unilaterally increasing his/her action by $\Delta = 1 - \sum_i s_i$. If $\sum_i s_i = 1$ then s is a NE – no player can unilaterally get a larger payoff. If $\sum_i s_i > 1$ then s is a NE if and only if no player can select a different action to get a strictly positive payoff, with is equivalent to the condition $|s_{-i}| \geq 1$ for all i , where $|s_{-i}|$ denotes the sum of the actions of all players except for player i .

4. **[Agreeing to disagree]**

Suppose $n \geq 3$ and consider the normal form game with n players, $I = [n]$, placed in a circle in order of index with wrap-around, so player n is next to players $n-1$ and 1. Let $S_i = \{0, 1\}$; each player i declares a bit s_i . Let $u_i(s_i) = \begin{cases} 1 & \text{if } s_{i-1} = s_{i+1} \neq s_i \\ 0 & \text{else} \end{cases}$ for $i \in I$, where, by notational convention, $s_0 = s_n$ and $s_{n+1} = s_1$.

- (a) Find a simple rule to determine whether a given strategy profile $s = (s_1, \dots, s_n)$ is a Nash equilibrium. Justify your answer.

Solution: A run for s is a set of consecutive indices (modulo n) such that the bits of s are identical. We claim that s is a Nash equilibrium if and only if all runs have length one or two. If s is a Nash equilibrium it cannot have a run of length three or more, or else one of the players not at the endpoints of the run could increase his/her payoff by switching actions. If all runs have length one or two, then if a player is in a run of length one, it cannot increase his/her payoff by switching, because the payoff is already at the maximum, namely one. If the player is in a run of length 2, the payoff could not increase his/her payoff by switching actions – such a switch would leave the payoff at zero.

- (b) Give an example of a Nash equilibrium in non-degenerate mixed strategies. Justify your answer.

Solution: Let $\sigma = (\sigma_i)_{i \in I}$ where $\sigma_i = (0.5, 0.5)$ for each i . In other words, σ is the profile such that each player selects an action by a fair coin flip. We claim that σ is a NE. It is true because for other players using σ_{-i} , the expected payoff of player i is $1/4$ for either action. So σ_i is a best response to σ_{-i} for all i .

5. **[True or false]**

Show the following statement is true, or show it is false. If a player in a normal form finite game has a dominant strategy, then the player must play that strategy with probability one for any correlated equilibrium.

Solution: True. Suppose p is a correlated equilibrium, which by definition means it is a probability distribution over the set of strategy profiles $S = S_1 \times \dots \times S_n$ such that for every player i and every strategy s'_i for player i ,

$$\sum_{s_{-i} \in S_{-i}} p(s_i, s_{-i})(u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i})) \geq 0. \quad (1)$$

Suppose s'_i is a dominant strategy for some player i , so that $u_i(s_i, s_{-i}) - u_i(s'_i, s_{-i}) < 0$ for all s_i with $s_i \neq s'_i$ and any choice of s_{-i} . Then for any strategy s_i for player i with $s_i \neq s'_i$, every term in the sum in (1) is less than or equal to zero, and is strictly less than zero if $p(s_i, s_{-i}) > 0$. Hence $p(s_i, s_{-i}) = 0$ for all s_{-i} . Thus, player i uses s_i with probability zero for any $s_i \neq s'_i$. Since the game is finite, player i thus must use s'_i with probability one.

6. [Guessing 2/3 of the average]

Consider the following game for n players. Each of the players selects a number from the set $\{1, \dots, 100\}$, and a cash prize is split evenly among the players whose numbers are closest to two-thirds the average of the n numbers chosen.

- (a) Show that the problem is solvable by iterated elimination of *weakly* dominated strategies, meaning the method can be used to eliminate all but one strategy for each player, which necessarily gives a Nash equilibrium. (A strategy μ_i of a player i is called weakly dominated if there is another strategy μ'_i that always does at least as well as μ_i , and is strictly better than μ_i for some vector of strategies of the other players.)

Solution: Any choice of number in the interval $\{68, \dots, 100\}$ is weakly dominated, because replacing a choice in that interval by the choice 67 (here 67 is $(2/3)100$ rounded to the nearest integer) would not cause a winning player to lose, while, for some choices of the other players, it could cause a losing player to win. Thus, after one step of elimination, we assume all players select numbers in the interval $\{1, \dots, 67\}$. After two steps of elimination we assume players select numbers in the set $\{1, \dots, 45\}$. After three steps, $\{1, \dots, 30\}$, and so on. At each step the set of remaining strategies has the form $\{1, \dots, k\}$, and as long as $k \geq 2$ the set shrinks at the next step. So the procedure terminates when all players choose the number one.

- (b) Give an example of a two player game, with two possible actions for each player, such that iterated elimination of weakly dominated strategies can eliminate a Nash equilibrium. (Hint: The eliminated Nash equilibrium might not be very good for either player.)

Solution: A bimatrix game with $A_1 = A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ gives such an example. Playing 2 is weakly dominated for each player, and eliminating those choices leads to the Nash equilibrium $(1, 1)$. However, $(2, 2)$ is also a Nash equilibrium.

- (c) Show that the Nash equilibrium found in part (a) is the unique mixed strategy Nash equilibrium (as usual we consider pure strategies to be special cases of mixed strategies). (Hint: Let k^* be the largest integer such that there exists at least one player choosing k^* with strictly positive probability. Show that $k^* = 1$.)

Solution: Consider a Nash equilibrium of mixed strategies. Let k^* be the largest integer such that there exists at least one player i choosing k^* with strictly positive probability. To complete the proof, we show that $k^* = 1$, meaning all players always choose the number one. For the sake of argument by contradiction, suppose $k^* \geq 2$. Let player i denote a player that plays k^* with positive probability. For any choice of strategies of

other players, player i has a pure strategy with a strictly positive probability of winning. Since k^* must be a best response for player i , it must therefore also have a strictly positive probability of winning. It is impossible for player i to win if no other chosen numbers are equal to k^* . (Indeed, if player i were the only one to choose k^* , the second highest chosen number would be strictly closer to $2/3$ of the average than k^* .) Thus, at least one of the other players must have a strictly positive probability of choosing k^* . But this means that player i could strictly increase her payoff by selecting $k^* - 1$ instead of k^* (indeed, such change would never change her from winning to losing, and in case she wins, she would win strictly more with positive probability) which contradicts the requirement that k^* be a best response for player i . This completes the argument by contradiction.