University of Illinois **Fall 2018**

ECE 586GT: Problem Set 6
Optimal selling mechanisms, and cooperative games

Due: Thursday, December 6, at beginning of class
Reading: Course notes, Sections 6.2 and Chapter 7.

1. [**A comparison of seller mechanisms with free disposal**]
Consider a seller with a single item to sell to two bidders. The private valuation X_i of the item to bidder i is known by the seller to be exponentially distributed with parameter λ_i for $i \in \{1, 2\}$. The bidders know their own valuations. The seller has zero value for the item (free disposal) and is not required to sell the item. For each of the following seller mechanisms, describe the allocation and payment rules, and find the expected payoff of the seller. The mechanisms should be individually rational and incentive compatible.

   (a) Myerson's mechanism that maximizes the seller's expected payoff. Assume the min-to-win form of payment rule. (Hint: The area rule for expectations could be useful: $\mathbb{E}[Y] = \int_0^\infty \mathbb{P}\{Y > t\}\, dt - \int_{-\infty}^0 \mathbb{P}\{Y \le t\}\, dt$.)

   (b) Vickrey second price auction with a reserve price r. You don't need to find a closed form expression for the choice of r that maximizes the expected reward. However, for part (d) the optimal threshold should be used but it can be found numerically. As mentioned above, if the item is not sold, the value to the seller is zero, even if $r \neq 0$.

   (c) A sequential selling mechanism. The seller first takes a bid from bidder 1 and, depending on the bid, either sells the object to bidder 1 for some price, or does not sell to bidder 1. In the later case, the seller takes a bid from bidder 2 and, based on the bid, either sells for a price, or does not sell. Your mechanism should maximize the seller's expected payoff over all such sequential selling mechanisms. (Hint: Work backwards.)

   (d) (This part will be worth only 2 points, so skip if it you are not interested in programming.) Let R_{opt}, R_v, and R_{12} denote the expected seller's payoff for the mechanisms of (a)-(c) respectively, and let R_{21} denote the expected payoff for part (c) if, instead, bidder 2 participates first. Plot all four expected payoffs in one plot vs. $E[X_2]$ (equal to $1/\lambda_2$) for $0 < E[X_2] \le 1$ and $E[X_1] = 1$.

2. [**A comparison of seller mechanisms when seller must sell.**]
Repeat problem 1 under the assumption that the seller must sell the object. In other words, the value of the object to the seller is $-\infty$. Let R_v^o, R_{opt}^o, R_{12}^o, and R_{21}^o denote the respective expected payoffs. For the payoff optimal mechanism, assume that $\lambda_1 \le \lambda_2$. (Hint: In part (a) you should find, for $\lambda_1 \le \lambda_2$, that $R_{opt}^o = e^{-1+\lambda_1/\lambda_2}\left[\frac{1}{\lambda_1} - \frac{1}{\lambda_1+\lambda_2}\right]$.)

3. [**The core of a flow game**]
We first review the max flow min cut theorem. A flow network (V, E, c, s, t) is a directed graph with vertex set V, edge set E, nonnegative edge capacities $c = (c_e : e \in E)$, and distinct vertices $s, t \in V$ representing the source and terminus of flow. Given a vector $(f_e)_{e \in E}$, the net flow into a vertex v is $\sum_{e \in E_{in}(v)} f_e - \sum_{e \in E_{out}(v)} f_e$. An $s-t$ flow is a vector $(f_e)_{e \in E}$ satisfying the capacity constraints $0 \le f_e \le c_e$ for all $e \in E$ such that the net flow into v is zero for any $v \in V \backslash \{s, t\}$. The value of an $s-t$ flow is the net flow into vertex t. An $s-t$ cut is a partition

$C = (V_1, V_2)$ of V with $s \in V_1$ and $t \in V_2$ and the capacity of such a cut is $\sum_{e \in X_C} c_e$, where X_C, the cutset of C, is the set of edges in E going from a vertex in V_1 to a vertex in V_2. The value of any flow is less than or equal to the capacity of any cut (weak duality) and the celebrated max-flow min-cut theorem is that strong duality holds. It is an instance of strong duality for linear programming (e.g. Problem 1, problem set 2, Fall 2017.)

Given a flow network (V, E, e, s, t) and a partition of E into disjoint sets $(E_i)_{i \in I}$ (written $E = \sqcup_{i \in I} E_i$), the flow game (I, v) is the cooperative game with set of players I and value function v such that for $S \subset I$, $v(S)$ is the maximum flow value for the network $(N, \sqcup_{i \in S} E_i, c, s, t)$. Show that the core of a flow game (I, v) is nonempty. (Hint: Think about how a payoff vector can be associated with a minimum value cut for the network with link set E.)

4. **[Shoe market]**
Consider the following market with transferable utilities, where $1 \leq n_1 \leq n_2$. Agents are indexed by $I = \{0\} \cup I_1 \cup I_2$ where $I_1 = \{1, \ldots, n_1\}$ and $I_2 = \{n_1 + 1, \ldots, n_1 + n_2\}$. There are $\ell = 2$ divisible goods, namely, units of left shoe and units of right shoe (Ignore the discrete nature of shoes!) The endowments are $w_0 = (0, 0)$, $w_i = (1, 0)$ for $i \in I_1$, and $w_i = (0, 1)$ for $i \in I_2$. The utility functions are given by $f_0(z_0) = \min\{z_{0,0}, z_{0,1}\}$ and $f_i(z_i) \equiv 0$ for $i \neq 0$. In words, agent 0 has unit value per pair of shoes, and the other agents each have a unit of shoe. If $n_1 < n_2$, there is less left shoe than right shoe.

   (a) Find the core, the payoff vector(s) for all competitive equilibria, and the Shapley value for the special case $n_1 = 1$ and $n_2 = 2$.

   (b) Find the core and competitive equilibrium (Shapley value not needed) in case $1 \leq n_1 < n_2$. Provide a brief explanation–full details not needed. Does your answers seem fair to you if $n_1 = 999$ and $n_2 = 1000$.

   (c) Find the core and competitive equilibrium in case $n_1 = n_2 = n$ for some $n \geq 1$. Provide a brief explanation–full details not needed.