University of Illinois **Fall 2018**

ECE 586GT: Problem Set 5

Games with incomplete information, multistage games, VCG mechanisims

Due: Thursday, November 15, at beginning of class
Reading: Course notes, Section 4.3 and Chapters 5 & 6

1. **[Supporting a public good with incomplete information]**
Consider n players, any of whom could perform a job that would benefit all the players. The players don't necessarily give the job high enough priority to coordinate about it. For example, the players could be participating in some venture and the job is to make preparations to ensure safety in case of some unanticipated event. Suppose player i selects $a_i \in \{0,1\}$, where $a_i = 1$ indicates that the player does the job. Suppose each player derives value 1 if at least one player does the job. Suppose θ_i is the type of player i, which represents the cost for player i to do the job. So the payoff of player i for a given action profile $a = (a_1, \ldots, a_n)$ and θ_i is $u_i(a, \theta_i) = -\theta_i a_i + \mathbf{1}_{\{\sum_{i\in[n]} a_i \geq 1\}}$. Player i can use the value of θ_i to select a_i; the other players don't see θ_i. Suppose the θ_i's are mutually independent and uniformly distributed over $[0,1]$ for $i \in [n]$.
   (a) Identify all the Bayes-Nash equilibria in pure strategies for $n = 2$. (A pure strategy for player i is a mapping $s_i : [0,1] \to \{0,1\}$, used to map θ_i to an action a_i.)
   (b) Identify all the Bayes-Nash equilibria in pure strategies for $n = 3$. (This problem is similar for any $n \geq 3$, but for simplicity you may stick to $n = 3$.)

2. **[Repeated zero sum games]**
Consider infinite repeated play of a finite, two-player zero sum game with some discount factor δ. What conclusion can be drawn from Nash's realization theorem for such a game?

3. **[Wide world of subgame perfect equilibria for repeated games]**
Consider infinite repeated play of the prisoners dilema game, with payoff matrix

	C	D
C	1,1	-1,2
D	2,-1	0,0

Let $n \geq 1$. Find $\bar{\delta}(n) \in (0,1)$ so that for any $\delta \in [\bar{\delta}, 1)$ and any script of plays of length n, there is a subgame perfect equilibrium for the infinite repeated game with discount factor δ such that, when used, the players both follow the given script for the first n plays. (You don't need to find the minimum possible value.)

4. **[Repeated play for the Bertrand equilibrium problem]**
Suppose n players, for some $n \geq 2$, represent firms that can each produce a common divisible good at a cost c per unit of good. Suppose the action of each player is to declare a price p_i per unit of good. Suppose there is an aggregate demand of consumers such that if the lowest price offered by any player is p_{min} then the consumers purchase a total quantity $(a - p_{\min})_+$ of goods, where a is a constant with $a > c$, and they purchase an equal amount from each player offering the minimum price. The game is among the players offering prices; the consumers are not considered to be part of the game. As shown in ECE 586GT problem set 1, problem 4(a), 2017, p is a Nash equilibrium if and only if $p_{\min} = c$ and $p_i = c$ for at least two players. The resulting payoff vector is the all zero vector. Basically, the competition among the players drives the sum of payoffs to zero at Nash equilibrium.

(a) Let's see if positive payoffs can be sustained in the infinite repeated game with the Bertrand based game as a stage game, with weight $(1-\delta)\delta^{t-1}$ on stage t. To begin, fix $\bar{p}$ with $c < \bar{p} < \frac{a+c}{2}$, and consider the trigger strategy based on the script that every player bids price $\bar{p}$ in every stage. Players follow the script, with switching to playing c forever if any player deviated from the script in an earlier stage. Each player will receive payoff $(a-\bar{p})(\bar{p}-c)/n$ per stage. Find the minimum δ so that such strategy profile is subgame perfect.

(b) Identify the set of payoff vectors v in the Nash realization region for the infinite repeated game with stage game based on the Bertrand selling mechanism.

5. **[Locating a center by VCG mechanism.]**
Suppose there are n players indexed by $[n]$, such that each player i has a location $x_i \in \mathcal{C}$ for some set $\mathcal{C}$. The problem is to find a mechanism to select a center (i.e. central location), $y \in \mathcal{C}$, and specify payments the players need to make if they choose to influence the choice of y through a bidding process. Suppose each player i has a type, $\theta_i > 0$, that represents how important it is for player i to be near y. Let ℓ be a function such that $\ell : C \times C \to \mathbb{R}_+$. The locations x_i are known to the decision mechanism, and the variable θ_i is private information of player i for each i. Suppose the total loss of player i is given by $L_i = \theta_i \ell(x_i, y_i) + m_i$, where m_i is the amount of money charged to player i by the mechanism.

(a) Describe use of a VCG mechanism to seclect $y \in \mathcal{C}$ and determine the payments $(m_i)_{i\in[n]}$, based on bids and the known locations of the players. To be definite select the payment rule so that the minimum payment possible for any given player is 0. (Hint: Since the loss function of player i has the form $y \mapsto \theta_i \ell(x_i, y)$ and x_i is known to the mechanism, the family of loss functions is one dimensional, so the bids can be one dimensional. Here the loss functions play the usual role of valuation functions, but with players seeking to minimize losses instead of maximizing valuations.)

(b) Simplify your expressions in case $\mathcal{C} = \mathbb{R}^d$ for some $d \geq 1$, and ℓ is squared Euclidean distance: $\ell(x, y) = \|x - y\|^2$. (Hint: Express the payment m_i in terms of y_i^*, where y_i^* is the center that would be selected if player i dropped out.)

(c) Find numerical values of y and the payments for part (b) in case $d = 1$, $n = 4$, $\theta_i = x_i = i$ for $i \in [n]$. (Hint: In the end you should find that the payment of player 3 happens to be zero.)

6. **[VCG applied to combinatorial auction]**
Suppose a VCG mechanism is applied to sell the objects in $\mathcal{O} = \{a, b, c\}$ to three bidders, who have submitted the following bids:

$$\begin{array}{lll} v_1\{a\} = 12, & v_1\{a,b\} = 14 & \\ v_2\{b,c\} = 6, & v_2\{a,b,c\} = 12 & \\ v_3\{b\} = 2, & v_3\{c\} = 5, & v_3\{b,c\} = 6. \end{array}$$

Assume the value of any bundle to any bidder, if not listed above, is zero. Also, assume that for any bidder, at most one of the bidder's bids may be used. In particular, by bidding such that $v_3\{b,c\} < v_3\{b\} + v_3\{c\}$, bidder 3 communicates that b and c are partial substitutes for the bidder. Suppose also that the seller has a reserve price, or value, of 1, for each item that is not sold. Determine the assignment of objects to bidders and the payments of the bidders, for the VCG mechanism under truthful bidding. Use the individually rational version of payments.